PROOF OF FORMULA 4.262.2

$$\int_0^1 \frac{\ln^3 x \, dx}{1-x} = -\frac{\pi^4}{15}$$

Expand the term $1/(1-x)$ in a geometric series to obtain

$$\int_0^1 \frac{\ln^3 x \, dx}{1-x} = \sum_{j=0}^{\infty} \int_0^1 x^j \ln^3 x \, dx.$$

The change of variables $t = -\ln x$ yields

$$\int_0^1 \frac{\ln^3 x \, dx}{1-x} = -\sum_{j=0}^{\infty} \int_0^{\infty} t^3 e^{-(j+1)t} \, dt.$$

Let $s = (j+1)t$ to obtain

$$\int_0^1 \frac{\ln^3 x \, dx}{1-x} = -\sum_{j=0}^{\infty} \frac{1}{(j+1)^4} \int_0^{\infty} s^3 e^{-s} \, ds.$$

The integral is $\Gamma(4) = 6$ and the series is $\zeta(4) = \pi^4/90$.