PROOF OF FORMULA 4.262.4

$$\int_0^1 \frac{x^n \ln^3 x}{1+x}\, dx = (-1)^{n+1} \left(\frac{7\pi^4}{120} - 6 \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4} \right)$$

The change of variables $x = e^{-t}$ gives

$$\int_0^1 \frac{x^n \ln^3 x}{1+x}\, dx = -\int_0^\infty \frac{t^3 e^{-(n+1)t}}{1+e^{-t}}\, dt.$$

Exopanding the integrand in a geometric series yields

$$\int_0^1 \frac{x^n \ln^3 x}{1+x}\, dx = -\sum_{k=0}^\infty (-1)^k \int_0^\infty t^3 e^{-(n+1+k)t}\, dt.$$

The change $u = (n+1+k)t$ shows that the integral is

$$6(-1)^n \sum_{j=n+1}^\infty \frac{(-1)^j}{j^4} = 6(-1)^n \left(\sum_{j=1}^\infty \frac{(-1)^j}{j^4} - \sum_{j=1}^n \frac{(-1)^j}{j^4} \right)$$

and the result follows from

$$\sum_{j=1}^\infty \frac{(-1)^j}{j^4} = -\frac{7}{8}\zeta(4) = -\frac{7\pi^4}{720}.$$