PROOF OF FORMULA 4.262.7

$$\int_0^1 \ln^3 x \, \frac{1-x^{n+1}}{(1-x)^2} \, dx = -\frac{(n+1)\pi^4}{15} + 6 \sum_{k=1}^{n} \frac{n-k+1}{k^4}$$

Start with $\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$ and differentiate to obtain $\frac{1}{(1-x)^2} = \sum_{j=1}^{\infty} j x^{j-1}$. Therefore

$$\begin{aligned}
\frac{1-x^{n+1}}{(1-x)^2} &= \sum_{j=1}^{\infty} j x^{j-1} - \sum_{j=1}^{\infty} j x^{j+n} \\
&= \sum_{j=1}^{\infty} j x^{j-1} - \sum_{t=n+2}^{\infty} (t-n-1) x^{t-1} \\
&= \sum_{j=1}^{n+1} j x^{j-1} + (n+1) \sum_{t=n+2}^{\infty} x^{t-1}.
\end{aligned}$$

It follows that

$$\int_0^1 \ln^3 x \, \frac{1-x^{n+1}}{(1-x)^2} \, dx = \sum_{j=1}^{\infty} j \int_0^1 x^{j-1} \ln^3 x \, dx + (n+1) \sum_{t=n+2}^{\infty} \int_0^1 x^{t-1} \ln^3 x \, dx.$$

Now

$$\int_0^1 x^{j-1} \ln^3 x \, dx = -\int_0^{\infty} u^3 e^{-ju} \, du = -\frac{6}{j^4}.$$

Thus

$$\int_0^1 \ln^3 x \, \frac{1-x^{n+1}}{(1-x)^2} \, dx = -6 \sum_{j=1}^{n+1} \frac{1}{j^3} - 6(n+1) \sum_{t=1}^{\infty} \frac{1}{j^4} + 6(n+1) \sum_{t=1}^{n+1} \frac{1}{j^4}.$$

The result follows from

$$\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}.$$