PROOF OF FORMULA 4.271.10

$$\int_0^1 \frac{\ln^{2n-1} x \, dx}{1-x^2} = \frac{1}{2} \int_0^\infty \frac{\ln^{2n-1} x \, dx}{1-x^2} = \frac{1-2^{2n}}{4n} |B_{2n}| \pi^{2n}$$

The first identity follows by checking that the integral over $[1, \infty)$ is the same as that over $[0, 1]$ via $x \mapsto 1/x$.

To obtain the value, expand the integrand in the form

$$\int_0^1 \frac{\ln^{2n-1} x \, dx}{1-x^2} = \sum_{k=0}^\infty \int_0^1 x^{2k} \ln^{2n-1} x \, dx.$$

The change of variable $t = -\ln x$ gives

$$\int_0^1 \frac{\ln^{2n-1} x \, dx}{1-x^2} = -(2n-1)! \sum_{k=0}^\infty \frac{1}{(2k+1)^{2n}}.$$

The standard even-odd splitting in the sum and the identity

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!} \pi^{2n}$$

gives the result.