PROOF OF FORMULA 4.271.13

$$\int_0^1 (\ln x)^{2n+1} \frac{\cos 2\pi a - x}{1 - 2x\cos 2\pi a + x^2}\, dx = -(2n-1)! \sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{k^{2n+2}}$$

The proof is based on the expression for the Poisson kernel

$$\frac{1-x^2}{1-2x\cos(2\pi a)+x^2} = 1 + 2\sum_{k=1}^{\infty} x^k \cos(2\pi a k)$$

that yields

$$\int_0^1 \frac{f(x)\, dx}{1-2x\cos(2\pi a)+x^2} = \int_0^1 \frac{f(x)\, dx}{1-x^2} + 2\sum_{k=1}^{\infty} \cos(2\pi a k) \int_0^1 \frac{x^k f(x)\, dx}{1-x^2}.$$

Denote $b = \cos(2\pi a)$, then in the present case the function is

$$f(x) = b(\ln x)^{2n+1} - x(\ln x)^{2n+1}.$$

Every resulting integral has the form

$$\begin{aligned} F_j &= \int_0^1 \frac{x^j (\ln x)^{2n+1}\, dx}{1-x^2} \\ &= \frac{1}{2^{2n+2}} \sum_{r=0}^{\infty} \int_0^1 t^{(2r+j-1)/2} (\ln t)^{2n+1}\, dt, \end{aligned}$$

after the change $t = x^2$ and expanding the integrand as a geometric series. The change of variables $t = e^{-v}$ gives

$$F_j = -(2n+1)! \sum_{r=0}^{\infty} \frac{1}{(2r+1+j)^{2n+2}}.$$

Therefore,

$$\int_0^1 (\ln x)^{2n+1} \frac{\cos 2\pi a - x}{1-2x\cos 2\pi a + x^2}\, dx = bJ_0 - J_1 + 2\sum_{k=1}^{\infty} \cos(2\pi a k) \left[bJ_k - J_{k+1} \right].$$

Elementary manipulations reduce this form to the stated result.