PROOF OF FORMULA 4.271.2

$$\int_0^1 \frac{\ln^{2n-1} x \, dx}{1+x} = \frac{1-2^{2n-1}}{2n} \pi^{2n} |B_{2n}|$$

Consider the more general formula

$$I(a) := \int_0^1 \frac{\ln^a x \, dx}{1+x}.$$

Expand the denominator as a geometric series to obtain

$$I(a) = \sum_{j=0}^{\infty} (-1)^j \int_0^1 x^j \ln^a x \, dx.$$

The change of variables $u = -\ln x$ gives

$$I(a) = \sum_{j=0}^{\infty} (-1)^{j+a} \int_0^{\infty} u^a e^{-(j+1)u} \, du,$$

and $t = (j+1)u$ yields

$$I(a) = \sum_{j=0}^{\infty} \frac{(-1)^{j+a}}{(j+1)^a} \int_0^{\infty} t^a e^{-t} \, dt.$$

The integral is recognized as $\Gamma(a+1)$. The sum is simplified using the standard even-odd split for the zeta function. It follows that

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j^a} = -\frac{2^{a-1}-1}{2^{a-1}} \zeta(a).$$

Therefore

$$I(a) = \frac{(-1)^a (2^a - 1)}{2^a} \Gamma(a+1) \zeta(a+1).$$

The case considered here is $a = 2n-1$. The result is first given in terms of $\zeta(2n)$ and this is simplified via the basic identity

$$\zeta(2n) = \frac{2^{2n-1}}{(2n)!} \pi^{2n} |B_{2n}|.$$

This appears as **9.542.1**.