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The integrals in Gradshteyn and Ryzhik. Part 1: a family of logarithmic integrals

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Abstract. We present the evaluation of a family of logarithmic integrals. This provides a unified proof of several formulas in the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik.

1. Introduction

The values of many definite integrals have been compiled in the classical *Table of Integrals, Series and Products* by I. S. Gradshteyn and I. M. Ryzhik [**3**]. The table is organized like a phonebook: integrals that *look* similar are place close together. For example, **4.229.4** gives

$$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{u-1} dx = \psi(\mu)\Gamma(\mu), \tag{1.1}$$

for $\operatorname{Re}\mu > 0$, and **4.229.7** states that

$$\int_{\pi/4}^{\pi/2} \ln\ln\tan x \, dx = \frac{\pi}{2}\ln\left\{\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right\}. \tag{1.2}$$

In spite of a large amount of work in the development of this table, the latest version of [**3**] still contains some typos. For example, the exponent u in (1.1) should be μ. A list of errors and typos can be found in

`http://www.mathtable.com/errata/gr6_errata.pdf`

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The fact that two integrals are close in the table is not a reflection of the difficulty involved in their evaluation. Indeed, the formula (1.1) can be established by the change of variables $v = -\ln x$ followed by differentiating the classical gamma function

$$\Gamma(\mu) := \int_0^\infty t^{\mu-1} e^{-t}\, dt, \quad \operatorname{Re}\mu > 0, \tag{1.3}$$

with respect to the parameter μ. The function $\psi(\mu)$ in (1.1) is simply the logarithmic derivative of $\Gamma(\mu)$ and the formula has been checked. The situation is quite different for (1.2). This formula is the subject of the lovely paper [**6**] in which the author uses Analytic Number Theory to check (1.2). The ingredients of the proof are quite formidable: the author shows that

$$\int_{\pi/4}^{\pi/2} \ln\ln\tan x\, dx = \frac{d}{ds}\Gamma(s)L(s) \text{ at } s = 1, \tag{1.4}$$

where

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots \tag{1.5}$$

is the Dirichlet L-function. The computation of (1.4) is done in terms of the Hurwitz zeta function

$$\zeta(q, s) = \sum_{n=0}^\infty \frac{1}{(n+q)^s}, \tag{1.6}$$

defined for $0 < q < 1$ and $\operatorname{Re} s > 1$. The function $\zeta(q, s)$ can be analytically continued to the whole plane with only a simple pole at $s = 1$ using the integral representation

$$\zeta(q, s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-qt} t^{s-1}}{1 - e^{-t}}\, dt. \tag{1.7}$$

The relation with the L-functions is provided by employing

$$L(s) = 2^{-2s} \left(\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4}) \right). \tag{1.8}$$

The functional equation

$$L(1-s) = \left(\frac{2}{\pi} \right)^s \sin \frac{\pi s}{2} \Gamma(s)\, L(s), \tag{1.9}$$

and Lerch's identity

$$\zeta'(0, a) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}, \tag{1.10}$$

complete the evaluation. More information about these functions can be found in [**7**].

In the introduction to [**2**] we expressed the desire to establish *all* the formulas in [**3**]. This is a *nearly impossible task* as was also noted by a (not so) favorable review given in [**5**]. This is the first of a series of papers where we present some of these evaluations.

We consider here the family

$$f_n(a) = \int_0^\infty \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)}, \text{ for } n \geqslant 2 \text{ and } a > 0. \tag{1.11}$$

Special examples of f_n appear in [**3**]. The reader will find

$$f_2(a) = \frac{\pi^2 + \ln^2 a}{2(1+a)} \tag{1.12}$$

as formula **4.232.3** and

$$f_3(a) = \frac{\ln a \, (\pi^2 + \ln^2 a)}{3(1+a)} \tag{1.13}$$

as formula **4.261.4**. In later sections the persistent reader will find

$$\begin{aligned} f_4(a) &= \frac{(\pi^2 + \ln^2 a)^2}{4(1+a)} \\ f_5(a) &= \frac{\ln a \, (\pi^2 + \ln^2 a)(7\pi^2 + 3\ln^2 a)}{15(1+a)} \\ f_6(a) &= \frac{(\pi^2 + \ln^2 a)^2 (3\pi^2 + \ln^2 a)}{6(1+a)} \end{aligned}$$

as **4.262.3**, **4.263.1** and **4.264.3** respectively.

These formulas suggest that

$$h_n(b) := f_n(a) \times (1+a) \tag{1.14}$$

is a polynomial in the variable $b = \ln a$. The relatively elementary evaluation of $f_n(a)$ discussed here identifies this polynomial.

There are several classical results that are stated without proof. The reader will find them in [**1**] and [**2**].

2. The evaluation

The expression (1.11) for $f_n(a)$ can be written as

$$f_n(a) = \int_0^1 \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)} + \int_1^\infty \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)},$$

and the transformation $t = 1/x$ in the second integral yields

$$f_n(a) = \int_0^1 \frac{\ln^{n-1} x \, dx}{(x-1)(x+a)} + (-1)^n \int_0^1 \frac{\ln^{n-1} x \, dx}{(x-1)(1+ax)}.$$

The partial decomposition

$$\frac{1}{(x-1)(x+a)} = \frac{1}{1+a}\frac{1}{x-1} - \frac{1}{1+a}\frac{1}{x+a}$$

yields the representation

$$f_n(a) = \frac{1-(-1)^{n-1}}{1+a} \int_0^1 \frac{\ln^{n-1} x \, dx}{x-1} - \frac{1}{1+a} \int_0^1 \frac{\ln^{n-1} x \, dx}{x+a} + (-1)^{n-1} \frac{a}{1+a} \int_0^1 \frac{\ln^{n-1} x \, dx}{1+ax}.$$

The evaluation of these integrals require the *polylogarithm* function defined by

$$\text{Li}_m(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^m}. \tag{2.1}$$

This function is sometimes denoted by PolyLog$[m, x]$. Detailed information about the polylogarithm functions appears in [**4**].

Proposition 2.1. For $n \in \mathbb{N}$, $n \geqslant 2$ and $a > 1$ we have

$$\begin{aligned} \int_0^1 \frac{\ln^{n-1} x \, dx}{x-1} &= (-1)^n (n-1)! \, \zeta(n), \\ \int_0^1 \frac{\ln^{n-1} x \, dx}{x+a} &= (-1)^n (n-1)! \, \text{Li}_n(-1/a), \\ \int_0^1 \frac{\ln^{n-1} x \, dx}{1+ax} &= (-1)^n \frac{(n-1)!}{a} \, \text{Li}_n(-a). \end{aligned}$$

Proof. Simply expand the integrand in a geometric series. □

Corollary 2.2. The integral $f_n(a)$ is given by

$$f_n(a) = \frac{(-1)^n (n-1)!}{1+a} \left\{ \left[(1 - (-1)^{n-1} \right] \zeta(n) - \text{Li}_n\left(-\tfrac{1}{a}\right) + (-1)^{n-1} \text{Li}_n(-a) \right\}.$$

The reduction of the previous expression requires the identity

$$\text{Li}_\nu(z) = \frac{(2\pi)^\nu}{\Gamma(\nu)} e^{\pi i \nu/2} \zeta\left(1-\nu, \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right) - e^{\pi i \nu} \text{Li}_\nu(-1/z). \tag{2.2}$$

This transformation for the polylogarithm function appears in

`http://functions.wolfram.com/10.08.17.0007.01`

In the special case $z = -a$ and $\nu = n$, with $n \in \mathbb{N}$, $n \geqslant 2$, we obtain

$$(-1)^{n-1} \text{Li}_n(-a) - \text{Li}_n(-1/a) = \frac{(2\pi)^n}{n! \, i^n} B_n\left(\frac{\log a}{2\pi i} + \frac{1}{2}\right), \tag{2.3}$$

where $B_n(z)$ is the Bernoulli polynomial of order n. This family of polynomials is defined by their exponential generating function

$$\frac{te^{qt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(q) \frac{t^k}{k!}. \tag{2.4}$$

The classical identity

$$\zeta(1-k, q) = -\frac{1}{k} B_k(q), \text{ for } k \in \mathbb{N} \tag{2.5}$$

is used in (2.3). Therefore the result in Corollary 2.2 can be written as:

Corollary 2.3. The integral $f_n(a)$ is given by

$$f_n(a) = \frac{(-1)^n}{1+a} (n-1)! \left[1 + (-1)^n \right] \zeta(n) + \frac{(2\pi i)^n}{n(1+a)} B_n\left(\frac{\log a}{2\pi i} + \frac{1}{2}\right).$$

We now proceed to simplify this representation. The Bernoulli polynomials satisfy the addition theorem

$$B_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} B_j(x) y^{n-j}, \tag{2.6}$$

and the reflection formula

$$B_n(\tfrac{1}{2} - x) = (-1)^n B_n\left(\tfrac{1}{2} + x\right). \tag{2.7}$$

In particular $B_n(\frac{1}{2}) = 0$ if n is odd. For n even, one has

$$B_n(\tfrac{1}{2}) = (2^{1-n} - 1)B_n, \tag{2.8}$$

where B_n is the Bernoulli number $B_n(0)$. Thus, the last term in Corollary 2.3 becomes

$$B_n\left(\frac{\log a}{2\pi i} + \frac{1}{2}\right) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2^{1-2j} - 1) B_{2j} \left(\frac{\log a}{2\pi i}\right)^{n-2j}.$$

We have completed the proof of the following closed-form formula for $f_n(a)$:

Theorem 2.4. The integral $f_n(a)$ is given by

$$\begin{aligned} f_n(a) &= \frac{(-1)^n\,(n-1)!}{1+a}\left[1 + (-1)^n\right]\zeta(n) + \\ &+ \frac{1}{n(1+a)} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2^{2j} - 2)(-1)^{j-1} B_{2j} \pi^{2j} (\log a)^{n-2j}. \end{aligned}$$

Observe that if n is odd, the first term vanishes and there is no contribution of the *odd zeta values*. For n even, the first term provides a rational multiple of π^n in view of Euler's representation of the even zeta values

$$\zeta(2m) = \frac{(-1)^{m+1}\,(2\pi)^{2m} B_{2m}}{2(2m)!}. \tag{2.9}$$

The polynomial h_n predicted in (1.14) can now be read directly from this expression for the integral f_n. Observe that h_n has positive coefficients because the Bernoulli numbers satisfy $(-1)^{j-1}B_{2j} > 0$.

Note. The change of variables $t = \ln x$ converts $h_n(a)$ into the form

$$h_n(a) = \int_{-\infty}^{\infty} \frac{t^{n-1}\,dt}{(1 - e^{-t})(a + e^t)}. \tag{2.10}$$

The integrals $h_n(a)$ for $n = 2, \cdots, 5$ appear in [**3**] as **3.419.2**, $\cdots$, **3.419.6**. The latest edition has an error in the expression for this last value.

Conclusions. We have provided an evaluation of the integral

$$f_n(a) := \int_0^{\infty} \frac{\ln^{n-1} x\,dx}{(x-1)(x+a)}, \tag{2.11}$$

given by

$$n(1+a)f_n(a) = (-1)^n n!\,[1+(-1)^n]\,\zeta(n) + \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} (2^{2j}-2)(-1)^{j-1} B_{2j} \pi^{2j} (\log a)^{n-2j}. \tag{2.12}$$

Symbolic calculation. We now describe our attempts to evaluate the integral $f_n(a)$ using Mathematica 5.2. For a specific value of n, Mathematica is capable of producing the result in (2.12). The integral is returned unevaluated if n is given as a parameter.

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