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The integrals in Gradshteyn and Ryzhik. Part 12: Some logarithmic integrals

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Abstract. We present the evaluation of some logarithmic integrals. The integrand contains a rational function with complex poles. The methods are illustrated with examples found in the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik.

1. Introduction

The classical table of integrals by I. Gradshteyn and I. M. Ryzhik [**3**] contains many entries from the family

$$\int_0^1 R(x)\,\log x\,dx \tag{1.1}$$

where R is a rational function. For instance, the elementary integral **4.231.1**

$$\int_0^1 \frac{\log x\,dx}{1+x} = -\frac{\pi^2}{12}, \tag{1.2}$$

is evaluated simply by expanding the integrand in a power series. In [**1**], the first author and collaborators have presented a systematic study of integrals of the form

$$h_{n,1}(b) = \int_0^b \frac{\log t\,dt}{(1+t)^{n+1}}, \tag{1.3}$$

as well as the case in which the integrand has a single purely imaginary pole

$$h_{n,2}(a,b) = \int_0^b \frac{\log t\,dt}{(t^2+a^2)^{n+1}}. \tag{1.4}$$

The work presented here deals with integrals where the rational part of the integrand is allowed to have arbitrary complex poles.

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2. Evaluations in terms of polylogarithms

In this section we describe the evaluation of

$$g(a) = \int_0^1 \frac{\log x \, dx}{x^2 - 2ax + 1}, \tag{2.1}$$

under the assumption that the denominator has non-real roots, that is, $a^2 < 1$.

The first approach to the evaluation of $g(a)$ is based on the factorization of the quartic as

$$x^2 - 2ax + 1 = (x + r_1)(x + r_2), \tag{2.2}$$

where $r_1 = -a + i\sqrt{1 - a^2}$ and $r_2 = -a - i\sqrt{1 - a^2}$. The partial fraction expansion

$$\frac{1}{(x + r_1)(x + r_2)} = \frac{1}{r_2 - r_1} \left(\frac{1}{x + r_1} - \frac{1}{x + r_2} \right), \tag{2.3}$$

yields

$$g(a) = \frac{1}{r_2 - r_1} \int_0^1 \frac{\log x \, dx}{x + r_1} - \frac{1}{r_2 - r_1} \int_0^1 \frac{\log x \, dx}{x + r_2}. \tag{2.4}$$

These integrals are computed in terms of the *dilogarithm* function defined by

$$\text{PolyLog}[2, x] := - \int_0^x \frac{\log(1 - t)}{t} \, dt. \tag{2.5}$$

A direct calculaton shows that

$$\int \frac{\log x \, dx}{x + a} = \log x \log(1 + x/a) + \text{PolyLog}[2, -x/a], \tag{2.6}$$

and thus

$$\int_0^1 \frac{\log x \, dx}{x + a} = \text{PolyLog} \left[2, -\frac{1}{a} \right]. \tag{2.7}$$

It follows that

$$g(a) = \frac{1}{r_2 - r_1} \left(\text{PolyLog} \left[2, -\frac{1}{r_1} \right] - \text{PolyLog} \left[2, -\frac{1}{r_2} \right] \right). \tag{2.8}$$

Observe that the real integral $g(a)$ appears here expressed in terms of the polylogarithm of complex arguments.

EXAMPLE 2.1. The case $a = 1/2$ yields

$$\int_0^1 \frac{\log x \, dx}{x^2 - x + 1} = \frac{i}{\sqrt{3}} \left(\text{PolyLog} \left[2, (1 + i\sqrt{3})/2 \right] - \text{PolyLog} \left[2, (1 - i\sqrt{3})/2 \right] \right). \tag{2.9}$$

The polylogarithm function is evaluated using the representation

$$(1 + i\sqrt{3})/2 = e^{i\pi/3}, \tag{2.10}$$

to produce

$$\begin{aligned}\text{PolyLog}\left[2,(1+i\sqrt{3})/2\right] &= \sum_{k=1}^{\infty}\frac{\left[\frac{1}{2}(1+i\sqrt{3})\right]^k}{k^2}=\sum_{k=1}^{\infty}\frac{e^{i\pi k/3}}{k^2}\\ &= \sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi k}{3}\right)+i\sin\left(\frac{\pi k}{3}\right)}{k^2}.\end{aligned}$$

Similarly

$$\text{PolyLog}\left[2,(1-i\sqrt{3})/2\right] = \sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi k}{3}\right)-i\sin\left(\frac{\pi k}{3}\right)}{k^2},$$

and it follows that

$$\begin{aligned}\int_0^1\frac{\log x\,dx}{x^2-x+1} &= \frac{i}{\sqrt{3}}\left(\text{PolyLog}\left[2,(1+i\sqrt{3})/2\right]-\text{PolyLog}\left[2,(1-i\sqrt{3})/2\right]\right)\\ &= -\frac{2}{\sqrt{3}}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi k}{3}\right)}{k^2}.\end{aligned}$$

The function $\sin(\pi k/3)$ is periodic, with period 6, and repeating values

$$\tfrac{\sqrt{3}}{2},\tfrac{\sqrt{3}}{2},0,-\tfrac{\sqrt{3}}{2},-\tfrac{\sqrt{3}}{2},0.$$

Therefore

$$\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi k}{3}\right)}{k^2}=\frac{\sqrt{3}}{2}\left(\sum_{k=0}^{\infty}\frac{1}{(6k+1)^2}+\sum_{k=0}^{\infty}\frac{1}{(6k+2)^2}-\sum_{k=0}^{\infty}\frac{1}{(6k+4)^2}-\sum_{k=0}^{\infty}\frac{1}{(6k+5)^2}\right).$$

To evaluate these sums, recall the series representation of the *polygamma* function $\psi(x)=\Gamma'(x)/\Gamma(x)$, given by

$$\psi(x)=-\gamma-\frac{1}{x}+\sum_{k=1}^{\infty}\frac{x}{k(x+k)}. \tag{2.11}$$

Differentiation yields

$$\psi'(x)=-\sum_{k=0}^{\infty}\frac{1}{(x+k)^2}, \tag{2.12}$$

and we obtain

$$\sum_{k=0}^{\infty}\frac{1}{(6k+j)^2}=\frac{1}{36}\sum_{k=0}^{\infty}\frac{1}{(k+\frac{j}{6})^2}.$$

This provides the expression

$$\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi k}{3}\right)}{k^2}=\frac{\sqrt{3}}{72}\left(\psi'\left(\tfrac{1}{6}\right)+\psi'\left(\tfrac{2}{6}\right)-\psi'\left(\tfrac{4}{6}\right)-\psi'\left(\tfrac{5}{6}\right)\right). \tag{2.13}$$

The integral (2.9) is

$$\int_0^1 \frac{\log x\,dx}{x^2-x+1} = -\frac{1}{36}\left(\psi'\left(\tfrac{1}{6}\right)+\psi'\left(\tfrac{2}{6}\right)-\psi'\left(\tfrac{4}{6}\right)-\psi'\left(\tfrac{5}{6}\right)\right). \tag{2.14}$$

The identities

$$\psi(1-x) = \psi(x) + \pi\cot\pi x, \tag{2.15}$$

and

$$\psi(2x) = \frac{1}{2}\left(\psi(x)+\psi(x+\tfrac{1}{2})\right) + \log 2, \tag{2.16}$$

produce

$$\psi'\left(\tfrac{1}{6}\right) = 5\psi'\left(\tfrac{1}{3}\right) - \tfrac{4\pi^2}{3},\ \psi'\left(\tfrac{2}{3}\right) = -\psi'\left(\tfrac{1}{3}\right) + \tfrac{4\pi^2}{3},\ \psi'\left(\tfrac{5}{6}\right) = -5\psi'\left(\tfrac{1}{3}\right) + \tfrac{16\pi^2}{3}.$$

Replacing in (2.14) yields

$$\int_0^1 \frac{\log x\,dx}{x^2-x+1} = \frac{2\pi^2}{9} - \frac{1}{3}\psi'\left(\frac{1}{3}\right). \tag{2.17}$$

This appears as formula **4.233.2** in [**3**].

Note. The method described in the previous example evaluates logarithmic integrals in terms of the *Clausen function*

$$\text{Cl}_2(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{k^2}. \tag{2.18}$$

Note. An identical procedure can be used to evaluate the integrals **4.233.1**, **4.233.3**, **4.233.4** in [**3**] given by

$$\int_0^1 \frac{\log x\,dx}{x^2+x+1} = \frac{4\pi^2}{27} - \frac{2}{9}\psi'\left(\frac{1}{3}\right), \tag{2.19}$$

$$\int_0^1 \frac{x\,\log x\,dx}{x^2+x+1} = -\frac{7\pi^2}{54} + \frac{1}{9}\psi'\left(\frac{1}{3}\right), \tag{2.20}$$

and

$$\int_0^1 \frac{x\,\log x\,dx}{x^2-x+1} = \frac{5\pi^2}{36} - \frac{1}{6}\psi'\left(\frac{1}{3}\right), \tag{2.21}$$

respectively.

3. An alternative approach

In this section we present an alternative evaluation for the integral

$$g(a) = \int_0^1 \frac{\log x \, dx}{x^2 - 2ax + 1}, \tag{3.1}$$

based on the observation that

$$g(a) = \lim_{s \to 0} \frac{d}{ds} \int_0^1 \frac{x^s \, dx}{x^2 - 2ax + 1}. \tag{3.2}$$

The proof discussed here is based on the *Chebyshev* polynomials of the second kind $U_n(a)$, defined by

$$U_n(a) = \frac{\sin[(n+1)t]}{\sin t}, \tag{3.3}$$

where $a = \cos t$. The relation with the problem at hand comes from the generating function

$$\frac{1}{1 - 2ax + x^2} = \sum_{k=0}^{\infty} U_k(a) x^k. \tag{3.4}$$

This appears as **8.945.2** in [**3**].

Observe that

$$\int_0^1 \frac{x^s \, dx}{x^2 - 2ax + 1} = \sum_{k=0}^{\infty} U_k(a) \int_0^1 x^{k+s} dx = \sum_{k=0}^{\infty} \frac{U_k(a)}{k+s+1}.$$

It follows that

$$\int_0^1 \frac{\log x \, dx}{x^2 - 2ax + 1} = - \sum_{k=0}^{\infty} \frac{U_k(a)}{(k+1)^2}. \tag{3.5}$$

Replacing the trigonometric expression (3.3) for the Chebyshev polynomial, it follows that

$$\int_0^1 \frac{\log x \, dx}{x^2 - 2ax + 1} = - \frac{1}{\sin t} \sum_{k=0}^{\infty} \frac{\sin kt}{k^2} = - \frac{\mathrm{Cl}_2(t)}{\sin t}. \tag{3.6}$$

This reproduces the representation discussed in Section 2.

Note. The methods presented here give the value of (3.1) in terms of the dilogarithm function. The classical values

$$\mathrm{Cl}_2\left(\tfrac{\pi}{2}\right) = -\mathrm{Cl}_2\left(\tfrac{3\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \text{Catalan}, \tag{3.7}$$

are easy to establish. More sophisticated evaluations appear in [**5**]. These are given in terms of the Hurwitz zeta function

$$\zeta(s,q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^s}. \tag{3.8}$$

For instance, the reader will find

$$\mathrm{Cl}_2\left(\tfrac{2\pi}{3}\right) = \sqrt{3}\left(\frac{3^{-s}-1}{2}\zeta(2) + 3^{-s}\zeta(2,\tfrac{1}{3})\right), \tag{3.9}$$

and

$$\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right) = \sqrt{3}\left(\frac{3^{-s}-1}{2}\zeta(2) + 6^{-s}\left(\zeta(2,\tfrac{1}{6}) + \zeta(2,\tfrac{1}{3})\right)\right). \tag{3.10}$$

Note. Integrals of the form

$$\int_0^1 R(x) \log\log\frac{1}{x}\,dx \tag{3.11}$$

present new challeges. The reader will find some examples in [**4**]. The current version of Mathematica is able to evaluate

$$\int_0^1 \frac{x\,\log\log 1/x}{x^4+x^2+1}\,dx = \frac{\pi}{12\sqrt{3}}\left(6\log 2 - 3\log 3 + 8\log\pi - 12\log\Gamma(\tfrac{1}{3})\right), \tag{3.12}$$

but is unable to evaluate

$$\int_0^1 \frac{x\,\log\log 1/x}{x^4-\sqrt{2}x^2+1}\,dx = \frac{\pi}{8\sqrt{2}}\left(7\log\pi - 4\log\sin\tfrac{\pi}{8} - 8\log\Gamma(\tfrac{1}{8})\right). \tag{3.13}$$

4. Higher powers of logarithms

The method of the previous sections can be used to evaluate integrals of the form

$$\int_0^1 R(x)\log^p x\,dx, \tag{4.1}$$

where R is a rational function. The ideas are illustrated with the verification of formula **4.261.8** in [**3**]:

$$\int_0^1 \frac{1-x}{1-x^6}\,\log^2 x\,dx = \frac{8\sqrt{3}\pi^3 + 351\zeta(3)}{486}. \tag{4.2}$$

Define

$$J_1 = \int_0^1 \frac{\log^2 x\,dx}{1+x}, \qquad J_2 = \int_0^1 \frac{\log^2 x\,dx}{1-x+x^2},$$
$$J_3 = \int_0^1 \frac{x\,\log^2 x\,dx}{1-x+x^2}, \qquad J_4 = \int_0^1 \frac{\log^2 x\,dx}{1+x+x^2}.$$

The partial fraction decomposition

$$\frac{1-x}{1-x^6} = \frac{1}{3}\frac{1}{1+x} + \frac{1}{6}\frac{1}{1-x+x^2} - \frac{1}{3}\frac{x}{1-x+x^2} + \frac{1}{2}\frac{1}{1+x+x^2},$$

gives

$$\int_0^1 \frac{1-x}{1-x^6}\,\log^2 x\,dx = \frac{1}{3}J_1 + \frac{1}{6}J_2 - \frac{1}{3}J_3 + \frac{1}{2}J_4. \tag{4.3}$$

Evaluation of J_1. Consider first

$$\int_0^1 \frac{x^s}{1+x}\,dx = \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{k+s-1}\,dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+s}.$$

Differentiating twice with respect to s gives

$$J_1 = \int_0^1 \frac{\log^2 x\,dx}{1+x} = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3} = \frac{3}{2}\zeta(3). \tag{4.4}$$

Evaluations of J_2. Integrating the expansion

$$\frac{x^s\,dx}{x^2-2ax+1} = \sum_{k=0}^{\infty}\frac{U_k(a)}{s+k+1}, \tag{4.5}$$

and differentiating twice with respect to s yields

$$\int_0^1 \frac{x^s\,\log^2 x\,dx}{x^2-2ax+1} = 2\sum_{k=0}^{\infty}\frac{U_k(a)}{(s+k+1)^3}. \tag{4.6}$$

The value $s=0$ yields

$$\int_0^1 \frac{\log^2 x\,dx}{x^2-2ax+1} = 2\sum_{k=0}^{\infty}\frac{U_k(a)}{(k+1)^3}. \tag{4.7}$$

We conclude that

$$J_2 = 2\sum_{k=0}^{\infty}\frac{U_k(\frac{1}{2})}{(k+1)^3}. \tag{4.8}$$

The sequence $U_k(\frac{1}{2})$ is periodic of period 6 and values 1, 0, -1, -1, 0, 1. Therefore

$$J_2 = 2\sum_{k=1}^{\infty}\frac{1}{(6k+1)^3} - 2\sum_{k=1}^{\infty}\frac{1}{(6k+3)^3} - 2\sum_{k=1}^{\infty}\frac{1}{(6k+4)^3} + 2\sum_{k=1}^{\infty}\frac{1}{(6k+5)^3}. \tag{4.9}$$

This can be written as

$$J_2 = \frac{1}{108}\left(\sum_{k=1}^{\infty}\frac{1}{(k+1/6)^3} - \sum_{k=1}^{\infty}\frac{1}{(k+1/2)^3} - \sum_{k=1}^{\infty}\frac{1}{(k+2/3)^3} + \sum_{k=1}^{\infty}\frac{1}{(k+5/6)^3}\right).$$

Proceeding along the same lines of the previous argument, employing now the second derivative of the polygamma function yields

$$J_2 = \frac{10\pi^3}{81\sqrt{3}}. \tag{4.10}$$

The same type of analysis gives

$$\begin{aligned} J_3 &= \int_0^1 \frac{x\,\log^2 x\,dx}{1-x+x^2} = \frac{5\pi^3}{81\sqrt{3}} - \frac{2\zeta(3)}{3}, \\ J_4 &= \int_0^1 \frac{\log^2 x\,dx}{1+x+x^2} = \frac{81\pi^3}{81\sqrt{3}}. \end{aligned}$$

This completes the proof of (4.2).

The reader is invited to use the method developed here to verify

$$\int_0^1 \frac{1-x}{1-x^6} \log^4 x \, dx = \frac{32\sqrt{3}\pi^5 + 16335\zeta(5)}{1458}, \tag{4.11}$$

and

$$\int_0^1 \frac{1-x}{1-x^6} \log^6 x \, dx = \frac{7(256\sqrt{3}\pi^7 + 1327995\zeta(7))}{26244}. \tag{4.12}$$

Mathematica 6.2 is capable of producing these results.

The methods discussed here constitute the most elementary approach to the evaluations of logarithmic integrals. M. Coffey [**2**] presents some of the more advanced techniques required for the computation of integrals of the form

$$\int_0^1 R(x) \log^s x \, dx \tag{4.13}$$

for s real and R a rational function.

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