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The integrals in Gradshteyn and Ryzhik. Part 20: Hypergeometric functions

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Abstract. The table of Gradshteyn and Ryzhik contains many integrals that involve the hypergeometric function ${}_pF_q$. Some examples are discussed.

1. Introduction

The hypergeometric function defined by

$$
{}_pF_q\left(a_1,\, a_2, \cdots, a_p; b_1,\, b_2, \cdots, b_q; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!} \tag{1.1}
$$

includes, as special cases, many of the elementary special functions. For example,

$$
\begin{aligned}
\log(1+x) &= x \; {}_2F_1\left(1,\, 1; 2; -x\right) \\
\sin x &= x \; {}_0F_1\left(-; \tfrac{3}{2}; -x^2/4\right) \\
\cosh x &= \lim_{a,\, b \to \infty} {}_2F_1\left(a, b; \tfrac{1}{2}; x^2/4ab\right).
\end{aligned} \tag{1.2}
$$

The binomial theorem, for real exponent, can also be expressed in hypergeometric form as

$$
(1-x)^{-a} = {}_1F_0\left(a; -; x\right). \tag{1.3}
$$

The goal of this paper is to verify the integrals in [**3**] that involve this function. Due to the large number of entries in [**3**] that can be related to hypergeometric functions, the list presented here represents the first part of these. More entries will appear in a future publication.

The hypergeometric function satisfies a large number of identities. The reader will find in [**1**] the best introduction to the subject. Some elementary identities are

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described here in detail. For example, if one of the top parameters (the a_i) agrees with a bottom one (the b_i), the function reduces to one with lower indices. The identity

$$ {}_2F_1(a,b;a;x) = {}_1F_0(a;-;x). \tag{1.4}$$

illustrates this point. The binomial theorem identifies the latter as $(1-x)^{-a}$.

2. Integrals over [0, 1]

The first result is a representation of ${}_2F_1$ in terms of the *beta integral*

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt. \tag{2.1}$$

Proposition 2.1. The hypergeometric function ${}_2F_1$ is given by

$$ {}_2F_1\left(a,\, b;\, c;\, x\right) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\, dt. \tag{2.2}$$

Proof. Expand the term $(1-tx)^{-a}$ by the binomial theorem and integrate term by term. ☐

This representation appears as **3.197.3** in [**3**]. In order to simplify the replacing of parameters, this entry is also written as

$$\int_0^1 t^b(1-t)^c(1-tx)^a\, dt = B(b+1,c+1)\, {}_2F_1\left(-a,\, b+1;\, b+c+2;\, x\right). \tag{2.3}$$

This is one of the forms in which it will be used here: the integral being the object of primary interest.

Example 2.2. The special case $a=c=1$ in (2.2) appears as **3.197.10** in [**3**]:

$$\int_0^1 \frac{t^{b-1}\, dt}{(1-t)^b\,(1+tx)} = \frac{\pi}{\sin \pi b}(1+x)^{-b}. \tag{2.4}$$

The evaluation is direct. The identity (1.4) gives

$$ {}_2F_1(1,\, b;\, 1;\, -x) = (1+x)^{-b} \tag{2.5}$$

and then use $B(b,1-b) = \Gamma(b)\Gamma(1-b) = \pi/\sin \pi b$ to complete the evaluation.

Example 2.3. Introduce the index r by $r=a-b$ and take $c=b+r$ in (2.2). Then we have

$$\int_0^1 t^{b-1}(1-t)^{r-1}(1-tx)^{-b-r}\, dt = B(b,r)\, {}_2F_1\left(b+r,\, b;\, b+r;\, x\right) \tag{2.6}$$

The identity (1.4) reduces the previous evaluation to

$$\int_0^1 t^{b-1}(1-t)^{r-1}(1-tx)^{-b-r}\, dt = B(b,r)\,(1-x)^{-b}. \tag{2.7}$$

This appears as **3.197.4** in [**3**].

3. A linear scaling

In this section integrals obtained from the basic representation (2.3) by the change of variables $y = tp$. This produces

$$\int_0^p y^{b-1}(p-y)^{c-b-1}(p-xy)^{-a}\,dy = p^{c-a-1}B(b,c-b)\,{}_2F_1\left(a,\, b;\, c; x\right). \tag{3.1}$$

Example 3.1. The special case $c = b+1$ produces

$$\int_0^p y^{b-1}(p-xy)^{-a}\,dy = \frac{1}{b}p^{b-a}\,{}_2F_1\left(a,\, b;\, b+1; x\right), \tag{3.2}$$

where we have used $B(b,1) = 1/b$. In order to eliminate the factor p^{-a}, we choose $x = -pr$ to obtain

$$\int_0^p y^{b-1}(1+ry)^{-a}\,dy = \frac{1}{p}u^{p}\,{}_2F_1\left(a,\, b;\, b+1; -rp\right), \tag{3.3}$$

This appears as **3.194.1** in [**3**]. The special case $a = 1$, stating that

$$\int_0^p \frac{y^{b-1}\,dy}{1+ry} = \frac{1}{b}p^{b}\,{}_2F_1\left(1,\, b;\, b+1; -rp\right), \tag{3.4}$$

appears as **3.194.5** in [**3**].

Example 3.2. The table [**3**] contains the formula **3.196.1**:

$$\int_0^u (x+b)^{\nu}(u-x)^{\mu-1}\,dx = \frac{b^{\nu}\,u^{\mu}}{\mu}\,{}_2F_1\left[1, -\nu, 1+\mu, -\frac{u}{b}\right]. \tag{3.5}$$

We believe that it is a bad idea to have u and μ in the same formula, so we write this as

$$\int_0^a (x+b)^{\nu}(a-x)^{\mu-1}\,dx = \frac{b^{\nu}\,a^{\mu}}{\mu}\,{}_2F_1\left[1, -\nu, 1+\mu, -\frac{a}{b}\right]. \tag{3.6}$$

To prove this, we let $x = at$ to get

$$\int_0^a (x+b)^{\nu}(a-x)^{\mu-1}\,dx = b^{\nu}a^{\mu}\int_0^1 (1+at/b)^{\nu}(1-t)^{\mu-1}\,dt. \tag{3.7}$$

The integral representation (2.3) now gives the result.

4. Powers of linear factors

The hypergeometric function appears in the evaluation of integrals of the form

$$I = \int_a^b L_1(x)^{\mu-1}L_2(x)^{\nu-1}L_3(x)^{\lambda-1}\,dx \tag{4.1}$$

where L_j are linear functions and $L_1(a) = L_2(b) = 0$. For example, **3.198**:

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\left[ax+b(1-x)+c\right]^{-(\mu+\nu)}\,dx = (a+c)^{-\mu}(b+c)^{-\nu}B(\mu,\nu) \tag{4.2}$$

is reduced to the normal form (2.3) by writing

$$I = (b+c)^{-\mu-\nu} \int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1-rx)^{-(\mu+\nu)}\, dx \tag{4.3}$$

with $r = (b-a)/(b+c)$. Then (2.3) gives

$$I = (b+c)^{-\mu-\nu} B(\mu,\nu)\, {}_2F_1\left(\mu+\nu,\, \mu;\, \mu+\nu; \frac{b-a}{b+c}\right). \tag{4.4}$$

To produce the stated answer, simply observe the special value of the hypergeometric function

$$ {}_2F_1(a,\, b;\, a; z) = (1-z)^{-b}. \tag{4.5}$$

Similarly, the evaluation of **3.199**:

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu}\, dx = (b-a)^{\mu+\nu-1}(b-c)^{-\mu}(a-c)^{-\nu} B(\mu,\nu), \tag{4.6}$$

is reduced to the interval $[0,1]$ by $t = (x-a)/(b-a)$ and then the result follows from **3.198**.

The specific form of the answer is sometimes simplified due to a special relation of the parameters μ, ν and λ in (4.1). For example, in the evaluation of **3.197.11**:

$$\int_0^1 \frac{x^{p-1/2}\, dx}{(1-x)^p\,(1+qx)^p} = \frac{2}{\sqrt{\pi}} \Gamma\left(p+\tfrac{1}{2}\right)\Gamma(1-p)\, \cos^{2p}(\varphi) \frac{\sin((2p-1)\varphi)}{(2p-1)\sin(\varphi)}, \tag{4.7}$$

with $\varphi = \arctan\sqrt{q}$. The standard reduction of the integral to hypergeometric form is easy. Write

$$I = \int_0^1 x^{p-1/2}(1-x)^{-p}(1+qx)^{-p}\, dx \tag{4.8}$$

and use (2.3) to obtain

$$I = B(p+\tfrac{1}{2}, 1-p)\, {}_2F_1\left(p,\, p+\tfrac{1}{2};\, \tfrac{3}{2};\, -q\right). \tag{4.9}$$

To reduce the answer to the stated form, we employ **9.121.19**:

$$ {}_2F_1\left(\frac{n+2}{2},\, \frac{n+1}{2};\, \frac{3}{2};\, -\tan^2 z\right) = \frac{\sin nz\, \cos^{n+1} z}{n\, \sin z}.$$

The evaluation of **3.197.12**:

$$\int_0^1 \frac{x^{p-1/2}\, dx}{(1-x)^p\,(1-qx)^p} = \frac{\Gamma(p+\frac{1}{2})\Gamma(1-p)}{\sqrt{\pi}} \frac{\left[(1-\sqrt{q})^{1-2p} - (1+\sqrt{q})^{1-2p}\right]}{(2p-1)\sqrt{q}}. \tag{4.10}$$

is done in similar form. The reduction to

$$I = B(p+\tfrac{1}{2}, 1-p)\, {}_2F_1\left(p,\, p+\tfrac{1}{2};\, \tfrac{3}{2};\, q\right) \tag{4.11}$$

is direct from (2.3). The stated form now follows from **9.121.4**:

$$ {}_2F_1\left(-\tfrac{n-1}{2}, -\tfrac{n}{2}+1;\, \tfrac{3}{2};\, \tfrac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}}.$$

5. Some quadratic factors

The table [**3**] contains several entries of the form

$$I = \int_a^b Q_1(x)^{\mu-1} L_2(x)^{\nu-1} L_3(x)^{\lambda-1}\, dx \tag{5.1}$$

where $Q_1(x)$ is a quadratic polynomial and L_j are linear functions. These are discussed in this section.

Example 5.1. The first entry evaluated here is **3.254.1**

$$\int_0^a x^{\lambda-1}(a-x)^{\mu-1}(x^2+b^2)^{\nu}\, dx = b^{2\nu} a^{\lambda+\mu-1} B(\lambda,\mu) \times {}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; -\frac{a^2}{b^2}\right).$$

The conditions given in [**3**] are $\text{Re}\left(\frac{a}{b}\right) > 0$, $\lambda > 0$, $\text{Re}\,\mu > 0$. This entry appears as entry 186(10) of [**2**] as an example of the Riemann-Liouville transform

$$f(x) \mapsto \frac{1}{\Gamma(\mu)} \int_0^y f(x)(y-x)^{\mu-1}\, dx. \tag{5.2}$$

It is convenient to scale the formula, by the change of variables $x = at$, to the form

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1+c^2t^2)^{\nu}\, dt = B(\lambda,\mu)\, {}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; -c^2\right),$$

with $c = a/b$. The binomial theorem gives

$$(1+c^2t^2)^{\nu} = {}_1F_0(-\nu; -; -c^2t^2) = \sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!}(-1)^n c^{2n} t^{2n} \tag{5.3}$$

that produces

$$\begin{aligned}\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1+c^2t^2)^{\nu}\, dt &= \sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!}(-c^2)^n \int_0^1 t^{\lambda+2n-1}(1-t)^{\mu-1}\, dt \\ &= \sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!}(-c^2)^n B(\lambda+2n,\mu).\end{aligned}$$

Now write the beta term as

$$\begin{aligned} B(\lambda+2n,\mu) &= \frac{\Gamma(\lambda+2n)\,\Gamma(\mu)}{\Gamma(\lambda+2n+\mu)} \\ &= \Gamma(\mu)\frac{2^{\lambda+2n-1}\Gamma(\frac{\lambda}{2}+n)\Gamma(\frac{\lambda+1}{2}+n)}{2^{\lambda+2n+\mu-1}\Gamma(\frac{\lambda+\mu}{2}+n)\Gamma(\frac{\lambda+\mu+1}{2}+n)}\end{aligned}$$

where the duplication formula for the gamma function

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\tfrac{1}{2}) \tag{5.4}$$

has been employed. The relation $\Gamma(x+m) = (x)_m \Gamma(x)$ now yields

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1+c^2t^2)^{\nu}\,dt =$$

$$\frac{\Gamma(\mu)\Gamma(\frac{\lambda}{2})\Gamma(\frac{\lambda+1}{2})}{2^{\mu}\,\Gamma(\frac{\lambda+\mu}{2})\Gamma(\frac{\lambda+\mu+1}{2})}\, {}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; -c^2\right).$$

Now simplify the gamma factors to produce the result.

Example 5.2. The next entry contains a typo in the *7th*-edition of [**3**]. The correct version of **3.254.2** states that

$$\int_a^{\infty} x^{-\lambda}(x-a)^{\mu-1}(x^2+b^2)^{\nu}\,dx = \tag{5.5}$$

$$a^{\mu-\lambda+2\nu}B(\mu, \lambda-\mu-2\nu)\, {}_3F_2\left(-\nu, \frac{\lambda-\mu}{2}-\nu, \frac{1+\lambda-\mu}{2}-\nu; \frac{\lambda}{2}-\nu, \frac{1+\lambda}{2}-\nu; -\frac{b^2}{a^2}\right)$$

that follows directly from Example 5.1 by the change of variables $y = a^2/x$. It is convenient to scale this entry to the form

$$\int_1^{\infty} t^{-\lambda}(t-1)^{\mu-1}(t^2+c^2)^{\nu}\,dt = \tag{5.6}$$

$$B(\mu, \lambda-\mu-2\nu)\, {}_3F_2\left(-\nu, \frac{\lambda-\nu}{2}-\nu, \frac{1+\lambda-\mu}{2}-\nu; \frac{\lambda}{2}-\nu, \frac{1+\lambda}{2}-\nu; -c^2\right).$$

6. A single factor of higher degree

In this section we conside entries in [**3**] of the

$$I = \int_a^b H_1(x)^{\mu-1}L_2(x)^{\nu-1}L_3(x)^{\lambda-1}\,dx \tag{6.1}$$

where $H_1(x)$ is a polynomial of degree $h \geqslant 2$ and L_j are linear functions.

Example 6.1. Entry **3.259.2** of [**3**] states that

$$\int_0^a x^{\nu-1}(a-x)^{\mu-1}(x^m+b^m)^{\lambda}\,dx = b^{m\lambda}a^{\mu+\nu-1}B(\mu,\nu)$$

$$\times {}_{m+1}F_m\left(-\lambda, \frac{\nu}{m}, \frac{\nu+1}{m}, \cdots, \frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}, \frac{\mu+\nu+1}{m}, \cdots, \frac{\mu+\nu+m-1}{m}; -\frac{a^m}{b^m}\right).$$

The scaling $t = x/a$ transforms this entry into

$$\int_0^1 t^{\nu-1}(1-t)^{\mu-1}(1+c^mt^m)^{\lambda}\,dt = B(\mu,\nu)$$

$$\times {}_{m+1}F_m\left(-\lambda, \frac{\nu}{m}, \frac{\nu+1}{m}, \cdots, \frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}, \frac{\mu+\nu+1}{m}, \cdots, \frac{\mu+\nu+m-1}{m}; -c^m\right)$$

with $c = a/b$. This is established next using a technique developed by Euler in his proof of the integral representation of ${}_2F_1$.

Start with

$$I = \int_0^1 t^{\nu-1}(1-t)^{\mu-1}\left(c^m t^m + 1\right)^{\lambda} dt$$
$$= \int_0^1 t^{\nu-1}(1-t)^{\mu-1}\, {}_1F_0\left(-\lambda; -; -c^m t^m\right) dt$$

using the elementary identity (1.3). This gives

$$I = \int_0^1 t^{\nu-1}(1-t)^{\mu-1} \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!}\left(-c^m t^m\right)^n dt$$
$$= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!}\left(-c^m\right)^n \int_0^1 t^{\nu+mn-1}(1-t)^{\mu-1} dt.$$

The integral is recognized as a beta function value, therefore

$$\begin{aligned}
I &= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!}\left(-c^m\right)^n \frac{\Gamma(\nu+mn)\Gamma(\mu)}{\Gamma(\nu+mn+\mu)} \\
&= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!}\left(-c^m\right)^n \frac{\Gamma(m(\frac{\nu}{m}+n))\Gamma(\mu)}{\Gamma(m(\frac{\nu+\mu}{m}+n))} \\
&= \Gamma(\mu)\sum_{n=0}^{\infty} \frac{(-\lambda)_n(-c^m)^n}{n!} \frac{m^{m(\nu/m+n)-1/2}\Gamma(\frac{\nu}{m}+n)\cdots\Gamma(\frac{\nu+m-1}{m}+n)}{m^{m(\frac{\nu+\mu}{m}+m)-1/2}\Gamma(\frac{\nu+\mu}{m}+n)\cdots\Gamma(\frac{\nu+\mu+m-1}{m}+n)} \\
&= \frac{\Gamma(\mu)}{m^{\mu}} \frac{\Gamma(\frac{\nu}{m})\cdots\Gamma(\frac{\nu+m-1}{m})}{\Gamma(\frac{\nu+\mu}{m})\cdots\Gamma(\frac{\nu+\mu+m-1}{m})} \times \sum_{n=0}^{\infty} \frac{(-\lambda)_n(\frac{\nu}{m})_n\cdots(\frac{\nu+m-1}{m})_n}{(\frac{\nu+\mu}{m})_n\cdots(\frac{\nu+\mu+m-1}{m})} \frac{(-c^m)^n}{n!} \\
&= \frac{\Gamma(\mu)}{m^{\mu}} \frac{\Gamma(\frac{\nu}{m})\cdots\Gamma(\frac{\nu+m-1}{m})}{\Gamma(\frac{\nu+\mu}{m})\cdots\Gamma(\frac{\nu+\mu+m-1}{m})} \times \\
&\quad \times {}_{m+1}F_m(-\lambda, \frac{\nu}{m}, \ldots, \frac{\nu+m-1}{m}; \frac{\nu+\mu}{m}, \ldots, \frac{\nu+\mu+m-1}{m}; -c^m).
\end{aligned}$$

This is the evaluation presented in entry **3.259.2**.

7. Integrals over a half-line

This section considers integrals over a half-line that can be expressed in terms of the hypergeometric function.

Example 7.1. To write (3.3) as an integral over an infinite half-line, make the change of variables $w = 1/y$ to obtain

$$\int_{1/u}^{\infty} w^{a-b-1}(1+w/r)^{-a}\, dw = \frac{u^b r^a}{b}\, {}_2F_1\left(a,\, b;\, b+1; -ru\right), \tag{7.1}$$

Now replace u by $1/u$ and r by $1/r$ to produce

$$\int_{u}^{\infty} w^{a-b-1}(1+rw)^{-a}\, dw = \frac{1}{bu^b r^a}\, {}_2F_1\left(a,\, b;\, b+1; -\frac{1}{ru}\right). \tag{7.2}$$

Finally let $b = a - s$ to obtain

$$\int_u^\infty w^{s-1}(1+rw)^{-a}\,dw = \frac{1}{(a-s)u^{a-s}r^a}\,{}_2F_1\left(a,\, a-s;\, a-s+1; -\frac{1}{ru}\right). \tag{7.3}$$

This appears as **3.194.2** in [**3**].

Example 7.2. The change of variable $y = 1/t$ converts (2.3) into **3.197.6**:

$$\int_1^\infty y^{a-c}(y-1)^{c-b-1}(\alpha y - 1)^{-a}\,dy = \alpha^{-a}B(b, c-b)\,{}_2F_1\left(a,\, b;\, c; 1/\alpha\right) \tag{7.4}$$

where we have labelled $\alpha = 1/x$.

Example 7.3. The change of variables $y = t/(1-t)$ converts (2.3) into **3.197.5**:

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(1+\alpha y)^{-a}\,dy = B(b, c-b)\,{}_2F_1\left(a,\, b;\, c; 1-\alpha\right) \tag{7.5}$$

where we have labelled $\alpha = 1 - x$. If we now replace α by $1/\alpha$ we obtain

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(y+\alpha)^{-a}\,dy = \alpha^{a}B(b, c-b){}_2F_1\left(a,\, b;\, c;\, 1-1/\alpha\right). \tag{7.6}$$

Use the identity

$${}_2F_1\left(a,\, b;\, c;\, 1-1/\alpha\right) = (1-\alpha)^a{}_2F_1\left(a,\, c-b;\, c;\, \alpha\right) \tag{7.7}$$

to produce **3.197.9**:

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(y+\alpha)^{-a}\,dy = \alpha^{a}B(b, c-b){}_2F_1\left(a,\, c-b;\, c;\, 1-\alpha\right). \tag{7.8}$$

Example 7.4. The change of variables $y = tu$ converts (2.3), with $-x$ instead of x, into **3.197.8**:

$$\int_0^u y^{b-1}(u-y)^{c-b-1}(y+\alpha)^{-a}\,dy = \alpha^{-a}u^{c-1}B(b, c-b)\,{}_2F_1\left(a,\, b;\, c; -u/\alpha\right) \tag{7.9}$$

where we have labelled $\alpha = u/x$.

Example 7.5. The change of variables $y = st/(1-t)$ converts (2.3) into

$$\int_0^\infty y^{b-1}(y+s)^{a-c}(y+r)^{-a}\,dy = r^{-a}s^{a+b-c}B(b, c-b){}_2F_1\left(a,\, b;\, c;\, 1-\frac{s}{r}\right), \tag{7.10}$$

where $r = s/(1-x)$. This is **3.197.1** in [**3**]. The special case $a = c - 1$ produces **3.227.1**:

$$\int_0^\infty \frac{y^{b-1}(y+r)^{1-c}}{y+s}\,dy = r^{1-c}s^{b-1}B(b, c-b){}_2F_1\left(c-1,\, b;\, c;\, 1-\frac{s}{r}\right). \tag{7.11}$$

Example 7.6. Now shift the lower limit of integration via $x = y + u$ to produce

$$\int_u^\infty (x-u)^{b-1}(x-u+s)^{a-c}(x-u+r)^{-a}\,dx = r^{-a}u^{a+b-c}B(b, c-b){}_2F_1\left(a,\, b;\, c;\, 1-\frac{s}{r}\right).$$

Choose $s = u$ and introduce the parameter v by $v = r - u$ to get

$$\int_u^\infty x^{a-c}(x-u)^{b-1}(x+v)^{-a}\,dx = (v+u)^{-a}u^{a+b-c}B(b,c-b)\,{}_2F_1\left(a,\,b;\,c;\,\frac{v}{v+u}\right).$$

Introduce new parameters via $a = -p$, keeping b and $c = q - p$. This yields

$$\begin{aligned}\int_u^\infty x^{-q}(x-u)^{b-1}(x+v)^{p}\,dx &= (v+u)^{p}u^{b-q}B(b,c-b-p)\,{}_2F_1\left(-p,\,b;\,q-p;\,\frac{v}{v+u}\right)\\ &= (v+u)^{p}u^{b-q}B(b,c-b-p)\,{}_2F_1\left(b,\,-p;\,q-p;\,\frac{v}{v+u}\right)\end{aligned}$$

where the symmetry of the hypergeometric function in its two variables has been used.

This result is transformed using **9.131.1**:

$$ {}_2F_1\,(a,\,b;\,c;\,z\,) = (1-z)^{-a}\,{}_2F_1\,(a,\,c-b;\,c;\,z/(z-1)\,)\,, \tag{7.12}$$

that gives

$$\int_u^\infty x^{-q}(x-u)^{b-1}(x+v)^{p}\,dx = (v+u)^{b+p}u^{b-q}B(b,q-p-b)\,{}_2F_1\left(b,\,q;\,q-p;\,-\frac{v}{u}\right).$$

This is the form that is found in **3.197.2**.

8. An exponential scale

The change of variables $t = e^{-r}$ in (2.3) produces

$$ {}_2F_1\,(a,\,b;\,c;\,x) = \frac{1}{B(b,c-b)}\int_0^\infty e^{-br}(1-e^{-r})^{c-b-1}(1-xe^{-r})^{-a}\,dr. \tag{8.1}$$

The parameters are relabeled by $a = \rho$, $b = \mu, c = \nu + \mu, x = \beta$ to produce **3.312.3**:

$$\int_0^\infty (1-e^{-x})^{\nu-1}(1-\beta e^{-x})^{-\rho}e^{-\mu x}\,dx = B(\mu,\nu)\,{}_2F_1\,(\rho,\,\mu;\,\mu+\nu;\,\beta)\,. \tag{8.2}$$

9. A more challenging example

The evaluation of **3.197.7**

$$\int_0^\infty x^{\mu-1/2}(x+s)^{-\mu}(x+r)^{-\mu}\,dx = \sqrt{\pi}(\sqrt{r}+\sqrt{s})^{1-2\mu}\frac{\Gamma(\mu-1/2)}{\Gamma(\mu)} \tag{9.1}$$

requires some more properties of the hypergeometric function.

The scaling $x = rt$ produces

$$I = s^{-\mu}\sqrt{r}\int_0^\infty t^{\mu-1/2}(1+t)^{-\mu}(1+rt/s)^{\mu}\,dt \tag{9.2}$$

and using **3.197.5** we have

$$I = s^{-\mu}\sqrt{r}B\left(\mu+\tfrac{1}{2},\mu-\tfrac{1}{2}\right)\,{}_2F_1\left(\mu,\mu+\tfrac{1}{2},2\mu;\,z\right) \tag{9.3}$$

where $z = 1 - r/s$. To simplify this expression we employ the relation

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) &= \frac{(1-z)^{-\alpha}\,\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)} {}_2F_1(\alpha, \gamma-\beta; \alpha-\beta+1; \frac{1}{1-z}) + \\ &+ \frac{(1-z)^{-\beta}\,\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} {}_2F_1(\beta, \gamma-\alpha; \beta-\alpha+1; \frac{1}{1-z}) \end{aligned}$$

to produce

$$\begin{aligned} {}_2F_1\left(\mu, \mu+\tfrac{1}{2}, 2\mu;\, z\right) &= \frac{(1-z)^{-\mu}\Gamma(2\mu)\Gamma(1/2)}{\Gamma(\mu+1/2)\,\Gamma(\mu)} {}_2F_1\left(\mu, \mu-\tfrac{1}{2}\,\tfrac{1}{2};\, \frac{1}{1-z}\right) \\ &+ \frac{(1-z)^{-\mu-1/2}\Gamma(2\mu)\Gamma(-1/2)}{\Gamma(\mu-1/2)\,\Gamma(\mu)} {}_2F_1\left(\mu, \mu+\tfrac{1}{2}\,\tfrac{3}{2};\, \frac{1}{1-z}\right). \end{aligned}$$

The binomial theorem shows that

$$ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; \frac{z^2}{t^2}\right) = \frac{1}{2t^n}\left((t+z)^n + (t-z)^n\right), \tag{9.4}$$

that appears as **9.121.2** in [**3**]. Thus

$$ {}_2F_1\left(\mu,\, \mu-\tfrac{1}{2};\, \tfrac{1}{2};\, \frac{1}{1-z}\right) = \frac{1}{2(1-z)^{1/2-\mu}}\left((1+\sqrt{1-z})^{1-2\mu} + (-1+\sqrt{1-z})^{1-2\mu}\right).$$

Similarly, **9.121.4** states that

$$ {}_2F_1\left(-\frac{n-1}{2}, -\frac{n-2}{2}; \frac{3}{2}; \frac{z^2}{t^2}\right) = \frac{1}{2nzt^{n-1}}\left((t+z)^n - (t-z)^n\right), \tag{9.5}$$

to produce

$$ {}_2F_1\left(\mu,\, \mu-\tfrac{1}{2};\, \tfrac{3}{2};\, \frac{1}{1-z}\right) = \frac{1}{2(1-2\mu)(1-z)^{-\mu}}\left((1+\sqrt{1-z})^{1-2\mu} - (-1+\sqrt{1-z})^{1-2\mu}\right).$$

Replacing these values in (9.3) produces the result.

10. One last example: a combination of algebraic factors and exponentials

Entry **3.389.1** presents an analytic expression for the integral

$$ I := \int_0^a x^{2\nu-1}(a^2-x^2)^{\rho-1}e^{\mu x}dx. \tag{10.1}$$

The evaluation begins with an elementary scaling to obtain

$$\begin{aligned} I &= a^{2(\rho-1)} \int_0^1 x^{2\nu-1}(1-\frac{x^2}{a^2})^{\rho-1}e^{\mu x}dx \\ &= \tfrac{1}{2}a^{2\rho-1}\int_0^1 (ay^{1/2})^{2\nu-1}(1-y)^{\rho-1}e^{\mu a y^{1/2}}y^{-1/2}dy. \end{aligned}$$

Now use ${}_0F_0(;;x) = e^x$ to obtain

$$
\begin{aligned}
I &= \frac{a^{2\rho+2\nu-2}}{2} \int_0^1 y^{\nu-1}(1-y)^{\rho-1}\, {}_0F_0(;;\mu a y^{1/2})\, dy \\
&= \frac{a^{2\rho+2\nu-2}}{2} \int_0^1 y^{\nu-1}(1-y)^{\rho-1} \sum_{n=0}^{\infty} \frac{(\mu a y^{1/2})^n}{n!} dy \\
&= \frac{a^{2\rho+2\nu-2}}{2} \sum_{n=0}^{\infty} \frac{(\mu a)^n}{n!} \int_0^1 y^{\nu+n/2-1}(1-y)^{\rho-1} dy.
\end{aligned}
$$

The integral is now recognized as a beta value to conclude that

$$
\begin{aligned}
I &= \frac{a^{2\rho+2\nu-2}}{2} \sum_{n=0}^{\infty} \frac{(\mu a)^n}{n!} B(\nu+n/2,\rho) \\
&= \frac{a^{2\rho+2\nu-2}\Gamma(\rho)}{2} \sum_{n=0}^{\infty} \frac{(\mu a)^n}{n!} \frac{\Gamma(\nu+n/2)}{\Gamma(\nu+n/2+\rho)} \\
&= \frac{a^{2\rho+2\nu-2}\Gamma(\rho)\Gamma(\nu)}{2\Gamma(\nu+\rho)} \sum_{k=0}^{\infty} \frac{(\mu a)^{2k}(\nu)_k}{\Gamma(2k+1)(\nu+\rho)_k} + \frac{a^{2\rho+2\nu-2}\Gamma(\rho)}{2} \sum_{k=0}^{\infty} \frac{(\mu a)^{2k+1}\Gamma(\nu+k+1/2)}{(2k+1)!\Gamma(\nu+\rho+k+1/2)}
\end{aligned}
$$

and combining the gamma factors to produce the beta function yields

$$
\begin{aligned}
I = &\frac{1}{2} a^{2\rho+2\nu-2} B(\rho,\nu) \sum_{k=0}^{\infty} \frac{(\mu^2 a^2)^k (\nu)_k}{(2k)\Gamma(2k)(\nu+\rho)_k} + \\
&+ \frac{1}{2} a^{2\rho+2\nu-1} \mu \Gamma(\rho) \sum_{k=0}^{\infty} \frac{(\mu a)^{2k}}{\Gamma(2k+2)} \frac{(\nu+1/2)_k \Gamma(\nu+1/2)}{(\nu+\rho+1/2)_k \Gamma(\nu+\rho+1/2)}.
\end{aligned}
$$

This can be reduced to

$$
\begin{aligned}
2I = &a^{2\rho+2\nu-2} B(\rho,\nu) \sum_{k=0}^{\infty} \frac{(\nu)_k (\mu^2 a^2)^k}{(\nu+\rho)_k (2k)} \frac{2^{1-2k}\sqrt{\pi}}{\Gamma(k)\Gamma(k+1/2)} + \\
&+ a^{2\rho+2\nu-1} \mu B(\rho,\nu+1/2) \sum_{k=0}^{\infty} \frac{(\nu+1/2)_k}{(\nu+\rho+1/2)_k} \frac{(\mu^2 a^2)^k 2^{1-2(k+1)}\sqrt{\pi}}{\Gamma(k+1)\Gamma(k+\frac{3}{2})} \\
= &a^{2\rho+2\nu-2} B(\rho,\nu) \sum_{k=0}^{\infty} \frac{(\nu)_k}{(\nu+\rho)_k (\frac{1}{2})_k k!} \left(\frac{\mu^2 a^2}{4}\right)^k + \\
&+ a^{2\rho+2\nu-1} \mu B(\rho,\nu+1/2) \sum_{k=0}^{\infty} \frac{(\nu+1/2)_k}{(\nu+\rho+1/2)_k \left(\frac{3}{2}\right)_k} \left(\frac{\mu^2 a^2}{4}\right)^k \\
= &a^{2\rho+2\nu-2} B(\rho,\nu)\, {}_1F_2\left(\nu;\nu+\rho,\frac{1}{2};\frac{\mu^2 a^2}{4}\right) + \\
&+ a^{2\rho+2\nu-1} \mu B(\rho,\nu+1/2)\, {}_1F_2\left(\nu+1;\nu+\rho+1/2,\frac{3}{2};\frac{\mu^2 a^2}{4}\right).
\end{aligned}
$$

There are many other entries of [**3**] that can be evaluated in terms of hypergeometric functions. A second selection of examples is in preparation.

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