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The integrals in Gradshteyn and Ryzhik. Part 21: Hyperbolic functions

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Abstract. The table of Gradshteyn and Ryzhik contains a variety of definite integrals of elementary functions. In this paper proofs for some of the entries where the integrand contains hyperbolic functions are provided.

1. Introduction

The table of integrals [**1**] contains some entries giving definite integrals where the integrand contains the classical standard *hyperbolic functions*, defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}. \tag{1.1}$$

Some of these entries are verified in the present paper.

2. Some elementary examples

In the evaluation of **3.511.1** in [**1**]:

$$\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a}, \quad \text{for } a > 0, \tag{2.1}$$

the parameter a can be scaled out of the equation. Indeed, the change of variables $t = ax$ yields

$$\int_0^\infty \frac{dt}{\cosh t} = \frac{\pi}{2}. \tag{2.2}$$

This can be reduced to a rational integrand by the change of variables $s = e^t$ to obtain

$$\begin{aligned} \int_0^\infty \frac{dt}{\cosh t} &= 2\int_1^\infty \frac{ds}{s^2+1} \\ &= 2\left(\tan^{-1}(\infty) - \tan^{-1} 1\right) = \frac{\pi}{2}. \end{aligned}$$

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Actually, the change of variables $s = e^t$ produces the value of the indefinite integral:

$$\int \frac{dt}{\cosh t} = 2 \int \frac{ds}{s^2+1} \tag{2.3}$$

that leads to

$$\int \frac{dt}{\cosh t} = 2 \tan^{-1}(e^t). \tag{2.4}$$

This appears as **2.423.9**.

EXAMPLE 2.1. The second elementary example presented here appears as entry **3.527.15**

$$\int_0^\infty \frac{\tanh(x/2)\,dx}{\cosh x} = \ln 2. \tag{2.5}$$

The integral is written as

$$\int_0^\infty \frac{\tanh(x/2)\,dx}{\cosh x} = 2\int_0^\infty \frac{e^x-1}{e^x+1}\,\frac{e^x\,dx}{e^{2x}+1}, \tag{2.6}$$

and the change of variables $t = e^{-x}$ gives

$$\int_0^\infty \frac{\tanh(x/2)\,dx}{\cosh x} = 2\int_0^1 \frac{1-t}{(1+t)(1+t^2)}\,dt. \tag{2.7}$$

The result now comes from an elementary partial fraction decomposition.

3. An example that is evaluated in terms of the Hurwitz zeta function

Special cases of the evaluation

$$\int_0^\infty \frac{x^n\,dx}{\cosh(x^m)} = \frac{\Gamma(p)}{m\,2^{2p-1}}\left[\zeta\left(p,\tfrac{1}{4}\right) - \zeta\left(p,\tfrac{3}{4}\right)\right], \tag{3.1}$$

appear in [**1**]. Here $p = \frac{n+1}{m}$ and

$$\zeta(z,q) = \sum_{k=1}^\infty \frac{1}{(k+q)^z} \tag{3.2}$$

is the Hurwitz zeta function. To prove (3.1) simply write

$$\int_0^\infty \frac{x^n\,dx}{\cosh(x^m)} = 2\int_0^\infty \frac{x^n e^{-x^m}\,dx}{1+e^{-2x^m}} \tag{3.3}$$

and expand the integrand as a geometric series to produce

$$\begin{aligned} I &= 2\sum_{j=0}^\infty (-1)^j \int_0^\infty x^n e^{-(2j+1)x^m}\,dx \\ &= 2\sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p}\int_0^\infty t^n e^{-t^m}\,dt. \end{aligned}$$

The change of variables $u = t^m$ shows that

$$\begin{aligned} \int_0^\infty t^n e^{-t^m} \, dt &= \frac{1}{m} \int_0^\infty u^{p-1} e^{-u} \, du \\ &= \frac{1}{m} \Gamma(p). \end{aligned}$$

It follows that

$$I = \frac{2\Gamma(p)}{m} \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p}. \tag{3.4}$$

Now split the sum according to the parity of j:

$$\begin{aligned} \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p} &= \sum_{j=0}^\infty \frac{1}{(4j+1)^p} - \sum_{j=0}^\infty \frac{1}{(4j+3)^p} \\ &= 2^{-2p} \left(\zeta(p, \tfrac{1}{4}) - \zeta(p, \tfrac{3}{4}) \right). \end{aligned}$$

Thus,

$$\int_0^\infty \frac{x^n \, dx}{\cosh(x^m)} = \frac{\Gamma(p)}{m \, 2^{2p-1}} \left[\zeta\left(p, \tfrac{1}{4}\right) - \zeta\left(p, \tfrac{3}{4}\right) \right] = \frac{2\Gamma(p)}{m} \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^p} \tag{3.5}$$

as claimed.

Example 3.1. In the case $n = m = 1$, the parameter $p = 2$ and **3.521.2** is obtained:

$$\int_0^\infty \frac{x \, dx}{\cosh x} = 2G \tag{3.6}$$

where G is **Catalan's constant** defined by

$$G := \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^2}. \tag{3.7}$$

The change of variables $u = e^{-t}$ yields **4.231.12**:

$$\int_0^1 \frac{\ln u \, du}{1+u^2} = -G. \tag{3.8}$$

Example 3.2. The case $n = 0$, $m = 2$ yields $p = 1/2$ and **3.511.8**:

$$\int_0^\infty \frac{dx}{\cosh(x^2)} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}, \tag{3.9}$$

follows from $\Gamma(1/2) = \sqrt{\pi}$. This integral has been replaced in the last edition of [**1**] by the elementary entry

$$\int_0^\infty \frac{dx}{\cosh^2(x)} = 1. \tag{3.10}$$

Example 3.3. The case $n = -1/2$, $m = 1$ yields $p = 1/2$ and the evaluation of **3.523.12**:

$$\int_0^\infty \frac{dx}{\sqrt{x}\, \cosh x} = 2\sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}, \tag{3.11}$$

Example 3.4. The case $n = 1/2$, $m = 1$ yields $p = 3/2$ and **3.523.11**:

$$\int_0^\infty \frac{\sqrt{x}\, dx}{\cosh x} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{(2k+1)^3}}, \tag{3.12}$$

follows from $\Gamma(3/2) = \sqrt{\pi}/2$.

The evaluation of

$$\int_0^\infty \frac{x^n\, dx}{\sinh(x^m)} = \frac{2\Gamma(p)}{m} \sum_{j=0}^\infty \frac{1}{(2j+1)^p}, \tag{3.13}$$

with $p = (n+1)/m$ is done exactly as above. The identity

$$\sum_{j=0}^\infty \frac{1}{(2j+1)^p} = \frac{2^p - 1}{2^p} \sum_{j=0}^\infty \frac{1}{j^p} \tag{3.14}$$

yields

$$\int_0^\infty \frac{x^n\, dx}{\sinh(x^m)} = \frac{\Gamma(p)}{m} \frac{2^p - 1}{2^{p-1}} \zeta(p). \tag{3.15}$$

Example 3.5. The special case $m = 1$ gives $p = n+1$ and

$$\int_0^\infty \frac{x^n\, dx}{\sinh x} = \Gamma(n+1) \frac{2^{n+1} - 1}{2^n} \zeta(n+1). \tag{3.16}$$

This appears as **3.523.1** in [**1**]. In particular $n = 1$ gives **3.521.1**:

$$\int_0^\infty \frac{x\, dx}{\sinh x} = \frac{\pi^2}{4}. \tag{3.17}$$

This comes in the apparently more general form

$$\int_0^\infty \frac{x\, dx}{\sinh ax} = \frac{\pi^2}{4a^2}. \tag{3.18}$$

But this reduces to the case $a = 1$ by the change of variables $t = ax$.

Example 3.6. The special case $n = 2k - 1$ gives **3.523.2**:

$$\int_0^\infty \frac{x^{2k-1}\, dx}{\sinh x} = \frac{2^{2k} - 1}{2k} |B_{2k}| \pi^{2k} \tag{3.19}$$

using

$$\zeta(2k) = \frac{2^{2k-1} |B_{2k}|}{(2k)!} \pi^{2k}. \tag{3.20}$$

The values $B_4 = -1/30$, $B_6 = 1/42$ and $B_8 = 1/30$ give **3.523.6**:

$$\int_0^\infty \frac{x^3\,dx}{\sinh x} = \frac{\pi^4}{8}, \tag{3.21}$$

and **3.523.8**:

$$\int_0^\infty \frac{x^5\,dx}{\sinh x} = \frac{\pi^6}{4}, \tag{3.22}$$

and **3.523.10**:

$$\int_0^\infty \frac{x^7\,dx}{\sinh x} = \frac{17\,\pi^8}{16}. \tag{3.23}$$

4. A direct series expansion

Entry **3.523.3** states that

$$\int_0^\infty \frac{x^{b-1}\,dx}{\cosh ax} = \frac{2\Gamma(b)}{(2a)^b}\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^b}. \tag{4.1}$$

The change of variables $t = ax$ shows that the entry is equivalent to the special case $a = 1$:

$$\int_0^\infty \frac{t^{b-1}\,dt}{\cosh t} = \frac{\Gamma(b)}{2^{b-1}}\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^b}. \tag{4.2}$$

The proof of (4.2) is obtained by modifying the integrand and expanding in series

$$\int_0^\infty \frac{t^{b-1}e^{-t}\,dt}{1+e^{-2t}} = \sum_{k=0}^\infty (-1)^k \int_0^\infty t^{b-1}e^{-(2k+1)t}\,dt. \tag{4.3}$$

The result follows via the change of variables $u = (2k+1)t$.

EXAMPLE 4.1. In the special case $b = 2n+1$, with $n \in \mathbb{N}$, the evaluation takes the form

$$\int_0^\infty \frac{x^{2n}\,dx}{\cosh x} = 2(2n)!\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{2n+1}}. \tag{4.4}$$

The series is represented in terms of the Euler numbers E_{2n} via the classical expression

$$\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{\pi^{2n+1}\,|E_{2n}|}{(2n)!\,2^{2n+2}} \tag{4.5}$$

to obtain **3.523.4**

$$\int_0^\infty \frac{x^{2n}\,dx}{\cosh x} = \left(\frac{\pi}{2}\right)^{2n+1}|E_{2n}|. \tag{4.6}$$

The Euler number can be computed from the exponential generating function

$$\frac{1}{\cosh t} = \sum_{n=0}^\infty \frac{E_n}{n!}t^n. \tag{4.7}$$

The first few values are $E_0 = 1$, $E_2 = -1$, $E_4 = 5$ and $E_6 = 61$. This gives the entries **3.523.5**

$$\int_0^\infty \frac{x^2\,dx}{\cosh x} = \frac{\pi^3}{8}, \tag{4.8}$$

3.523.7

$$\int_0^\infty \frac{x^4\,dx}{\cosh x} = \frac{5\pi^5}{32}, \tag{4.9}$$

and **3.523.9**

$$\int_0^\infty \frac{x^6\,dx}{\cosh x} = \frac{61\pi^7}{128}. \tag{4.10}$$

5. An example involving Catalan constnat

Entry **3.527.14** states that

$$\int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x}\,dx = 4G, \tag{5.1}$$

where G is Catalan's constant defined in (3.7). The evaluation is obtained by writing the integral as

$$\int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x}\,dx = 2\int_0^\infty \frac{x^2(e^x - e^{-x})\,e^{-2x}}{(1+e^{-2x})^2}\,dx \tag{5.2}$$

and expanding in a geometric series to produce

$$\int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x}\,dx = -2\sum_{k=1}^\infty (-1)^k k \int_0^\infty x^2(e^x - e^{-x})e^{-2kx}\,dx. \tag{5.3}$$

Integrate term by term to obtain

$$\int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x}\,dx = -4\sum_{k=1}^\infty (-1)^k k \left[\frac{1}{(2k-1)^3} - \frac{1}{(2k+1)^3} \right]. \tag{5.4}$$

Simple manipulations of the last two series produce the result.

6. Quotients of hyperbolic functions

Section 3.5 of [**1**] contains several evaluations where the integrand contains quotients of hyperbolic functions. This section describes a selection of them.

Example 6.1. Formula **3.511.2** states that

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tan\frac{\pi a}{2b} \tag{6.1}$$

To evaluate this entry start with the change of variables $t = e^{-x}$ to obtain

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \int_0^1 \frac{t^{a-b-1} - t^{-a-b-1}}{1-t^{-2b}}\,dt \tag{6.2}$$

and continue with $u = t^{2b}$ to produce

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{1}{2b}\int_0^1 \frac{u^{-c-1/2} - u^{c-1/2}}{1-u}\,du \tag{6.3}$$

with $c = a/2b$. The evaluation of this last form employs formula **3.231.5** in [**1**]

$$\int_0^1 \frac{x^{\mu-1} - x^{\nu-1}}{1-x}\,dx = \psi(\nu) - \psi(\mu), \tag{6.4}$$

where $\psi(a) = \frac{d}{da}\ln\Gamma(a)$ is the logarithmic derivative of the gamma function. This formula was established in [**3**]. It follows that

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{1}{2b}\left(\psi(c+\tfrac{1}{2}) - \psi(-c+\tfrac{1}{2})\right). \tag{6.5}$$

The final form of the evaluation comes from the identity **8.365.9**

$$\psi(\tfrac{1}{2}+c) = \psi(\tfrac{1}{2}-c) + \pi\tan\pi c. \tag{6.6}$$

EXAMPLE 6.2. Differentiating (6.1) $2m$-times with respect to a yields **3.524.2**

$$\int_0^\infty x^{2m}\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m}}{da^{2m}}\left(\tan\frac{\pi a}{2b}\right), \tag{6.7}$$

with special cases **3.524.9**

$$\int_0^\infty x^2\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi^3}{4b^3}\sin\frac{\pi a}{2b}\sec^3\frac{\pi a}{2b},$$

3.524.10

$$\int_0^\infty x^4\frac{\sinh ax}{\sinh bx}\,dx = 8\left(\frac{\pi}{2b}\sec\frac{\pi a}{2b}\right)^5\cdot\sin\frac{\pi a}{2b}\cdot\left(2+\sin^2\frac{\pi a}{2b}\right),$$

and **3.524.11**

$$\int_0^\infty x^6\frac{\sinh ax}{\sinh bx}\,dx = 16\left(\frac{\pi}{2b}\sec\frac{\pi a}{2b}\right)^7\cdot\sin\frac{\pi a}{2b}\cdot\left(45-30\cos^2\frac{\pi a}{2b}+2\cos^4\frac{\pi a}{2b}\right).$$

An odd number of differentiations of (6.1) yields **3.524.8**

$$\int_0^\infty x^{2m+1}\frac{\cosh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m+1}}{da^{2m+1}}\left(\tan\frac{\pi a}{2b}\right), \tag{6.8}$$

with special cases **3.524.16**

$$\int_0^\infty x\,\frac{\cosh ax}{\sinh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{\pi a}{2b}\right)^2,$$

3.524.17

$$\int_0^\infty x^3\,\frac{\cosh ax}{\sinh bx}\,dx = 2\left(\frac{\pi}{2b}\sec\frac{\pi a}{2b}\right)^4\left(1+2\sin^2\frac{\pi a}{2b}\right),$$

3.524.18

$$\int_0^\infty x^5\,\frac{\cosh ax}{\sinh bx}\,dx = 8\left(\frac{\pi}{2b}\sec\frac{\pi a}{2b}\right)^6\left(15-15\cos^2\frac{\pi a}{2b}+2\cos^4\frac{\pi a}{2b}\right),$$

and **3.524.19**

$$\int_0^\infty x^7 \frac{\cosh ax}{\sinh bx}\, dx = 16 \left(\frac{\pi}{2b} \sec \frac{\pi a}{2b} \right)^8 \left(315 - 420 \cos^2 \frac{\pi a}{2b} + 126 \cos^4 \frac{\pi a}{2b} - 4 \cos^6 \frac{\pi a}{2b} \right).$$

EXAMPLE 6.3. Entry **3.511.4** states that

$$\int_0^\infty \frac{\cosh ax}{\cosh bx}\, dx = \frac{\pi}{2b} \sec \frac{\pi a}{2b}. \tag{6.9}$$

The proof follows the procedure employed in Example 6.1. The change of variables $u = e^{-2bx}$ gives

$$\int_0^\infty \frac{\cosh ax}{\cosh bx}\, dx = \frac{1}{2b} \int_0^1 \frac{u^{c-1/2} + u^{-c-1/2}}{1+u}\, du. \tag{6.10}$$

Now employ **3.231.2**

$$\int_0^1 \frac{x^{p-1} + x^{-p}}{1+x}\, dx = \frac{\pi}{\sin \pi p} \tag{6.11}$$

with $p = c + 1/2$. This integral was evaluated in [**2**].

EXAMPLE 6.4. Differentiating (6.9) an even number of times with respect to the parameter a gives **3.524.6** :

$$\int_0^\infty x^{2m} \frac{\cosh ax}{\cosh bx}\, dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \left(\sec \frac{\pi a}{2b} \right). \tag{6.12}$$

The special cases **3.524.20**

$$\int_0^\infty x^2 \frac{\cosh ax}{\cosh bx}\, dx = \frac{\pi^3}{8b^3} \left(2 \sec^3 \frac{\pi a}{2b} - \sec \frac{\pi a}{2b} \right),$$

3.524.21

$$\int_0^\infty x^4 \frac{\cosh ax}{\cosh bx}\, dx = \left(\frac{\pi}{2b} \sec \frac{\pi a}{2b} \right)^5 \left(24 - 20 \cos^2 \frac{\pi a}{2b} + \cos^4 \frac{\pi a}{2b} \right),$$

and **3.524.22**

$$\int_0^\infty x^6 \frac{\cosh ax}{\cosh bx}\, dx = \left(\frac{\pi}{2b} \sec \frac{\pi a}{2b} \right)^7 \left(720 - 840 \cos^2 \frac{\pi a}{2b} + 184 \cos^4 \frac{\pi a}{2b} - \cos^6 \frac{\pi a}{2b} \right),$$

are obtained by performing the differentiation.

EXAMPLE 6.5. Differentiating (6.9) an odd number of times with respect to the parameter a gives **3.524.4**

$$\int_0^\infty x^{2m+1} \frac{\sinh ax}{\cosh bx}\, dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \left(\sec \frac{\pi a}{2b} \right). \tag{6.13}$$

The special cases **3.524.12**

$$\int_0^\infty x \frac{\sinh ax}{\cosh bx}\, dx = \frac{\pi^2}{4b^2} \sin \frac{\pi a}{2b} \sec^2 \frac{\pi a}{2b},$$

3.524.13

$$\int_0^\infty x^3 \frac{\sinh ax}{\cosh bx}\, dx = \left(\frac{\pi}{2b} \sec \frac{\pi a}{2b} \right)^4 \sin \frac{\pi a}{2b} \left(6 - \cos^2 \frac{\pi a}{2b} \right),$$

3.524.14

$$\int_0^\infty x^5 \frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b} \sec \frac{\pi a}{2b} \right)^6 \sin \frac{\pi a}{2b} \left(120 - 60\cos^2 \frac{\pi a}{2b} + \cos^4 \frac{\pi a}{2b} \right),$$

and **3.524.15**

$$\int_0^\infty x^7 \frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b} \sec \frac{\pi a}{2b} \right)^8 \sin \frac{\pi a}{2b} \left(5040 - 4200\cos^2 \frac{\pi a}{2b} + 546\cos^4 \frac{\pi a}{2b} - \cos^6 \frac{\pi a}{2b} \right)$$

are obtained as before.

Example 6.6. Integrate (6.9) with respect to the parameter a produces

$$\int_0^\infty \frac{\sinh ax}{\cosh bx} \frac{dx}{x} = \ln \tan \left(\frac{\pi a}{4b} + \frac{\pi}{4} \right). \tag{6.14}$$

This appears as entry **3.524.23** in [**1**]. The evaluation employs the elementary primitive (that appears as entry **2.01.14**)

$$\int \sec u\, du = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right). \tag{6.15}$$

Example 6.7. Entry **3.527.6** states that

$$\int_0^\infty \frac{x^{\mu-1} \sinh ax}{\cosh^2 ax}\,dx = \frac{2\Gamma(\mu)}{a^\mu} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\mu-1}} \tag{6.16}$$

that can be scaled to the case $a = 1$ by $t = ax$

$$\int_0^\infty \frac{t^{\mu-1} \sinh t}{\cosh^2 t}\,dt = 2\Gamma(\mu) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\mu-1}}. \tag{6.17}$$

To evaluate this last form write the integrand as

$$\int_0^\infty \frac{t^{\mu-1} \sinh t}{\cosh^2 t}\,dt = 2\int_0^\infty t^{\mu-1}(e^t - e^{-t})e^{-2t} \frac{dt}{(1+e^{-2t})^2} \tag{6.18}$$

and expand it in a power series and integrate it to obtain

$$\int_0^\infty \frac{t^{\mu-1} \sinh t}{\cosh^2 t}\,dt = 2\Gamma(\mu) \left[1 + \sum_{k=1}^\infty \frac{(-1)^k (k+1)}{(2k+1)^\mu} - \sum_{k=1}^\infty \frac{(-1)^{k+1}\, k}{(2k+1)^\mu} \right]. \tag{6.19}$$

This is the right-hand side of (6.17).

The special case $\mu = 2$ and the series

$$\sum_{k=0}^\infty \frac{(-1)^k}{2k+1} = \frac{\pi}{4} \tag{6.20}$$

yield the evaluation of entry **3.527.7**

$$\int_0^\infty \frac{x \sinh x}{\cosh^2 x}\,dx = \frac{\pi}{2}. \tag{6.21}$$

The special case $\mu = 2m+2$ and the series for the Euler numbers in (4.5) produce the evaluation of entry **3.527.8**

$$\int_0^\infty \frac{x^{2m+1}\,\sinh x}{\cosh^2 x}\,dx = (2m+1)\left(\frac{\pi}{2}\right)^{2m+1}|E_{2m}|. \tag{6.22}$$

7. An evaluation by residues

Entry **3.522.3**

$$\int_0^\infty \frac{dx}{(b^2+x^2)\,\cosh ax} = \frac{2\pi}{b}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{2ab+(2k-1)\pi} \tag{7.1}$$

is now evaluated by the method of residues. The change of variables $t = bx$ shows that it suffices to evaluate this integral for $b = 1$; that is,

$$\int_0^\infty \frac{dx}{(1+x^2)\,\cosh ax} = 2\pi\sum_{k=1}^\infty \frac{(-1)^{k-1}}{2a+(2k-1)\pi}. \tag{7.2}$$

The integrand $f(x)$ is an even function, therefore the evaluation requested is equivalent to

$$\int_{-\infty}^\infty f(x)\,dx = \pi\sum_{k=1}^\infty \frac{(-1)^{k-1}}{2a+(2k-1)\pi}. \tag{7.3}$$

The integral is computed by closing the real axis with a semi-circle centered at the origin located in the upper half-plane. An elementary estimate shows that the integral over the circular boundary vanishes as the radius goes to infinity. Therefore,

$$\int_{-\infty}^\infty f(x)\,dx = 2\pi i\sum_{p_j}\text{Res}(f;p_j) \tag{7.4}$$

where p_j is a pole of f in the upper-half plane. The integrand has poles at $z = i$ and $z = \frac{(2k-1)\pi i}{2a}$ for $k \in \mathbb{N}$. The poles are simple, unless $(2k+1)\pi = 2a$ for some k. Aside from this special case, the residues are computed as

$$\begin{aligned}\text{Res}(f;i) &= \frac{1}{2i\cosh(ia)} = \frac{1}{2i\cos a}\\ \text{Res}\left(f;\frac{(2k-1)\pi i}{2a}\right) &= \frac{(-1)^{k-1}4ia}{4a^2-\pi^2(2k-1)^2}.\end{aligned}$$

The residue theorem and a partial fraction decomposition give the stated value of the integral.

Example 7.1. The special case $a = \pi$ and $b = 1$ gives

$$\int_0^\infty \frac{dx}{(1+x^2)\,\cosh \pi x} = 2\sum_{k=1}^\infty \frac{(-1)^{k-1}}{2k+1} \tag{7.5}$$

and

$$\sum_{k=0}^\infty \frac{(-1)^k}{2k+1} = \frac{\pi}{4} \tag{7.6}$$

provides entry **3.522.6**:

$$\int_0^\infty \frac{dx}{(1+x^2)\,\cosh \pi x} = 2 - \frac{\pi}{2}. \tag{7.7}$$

EXAMPLE 7.2. The special case $a = \pi/2$ and $b = 1$ gives

$$\int_0^\infty \frac{dx}{(1+x^2)\,\cosh \frac{\pi x}{2}} = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}. \tag{7.8}$$

The evaluation

$$\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} = \ln 2 \tag{7.9}$$

yields

$$\int_0^\infty \frac{dx}{(1+x^2)\,\cosh \frac{\pi x}{2}} = \ln 2. \tag{7.10}$$

This is entry **3.522.8**.

EXAMPLE 7.3. The choice $a = \pi/4$ and $b = 1$ gives

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(\pi x/4)} = 4\sum_{k=1}^\infty \frac{(-1)^{k-1}}{4k-1}. \tag{7.11}$$

Entry **3.522.10** states that

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(\pi x/4)} = \frac{1}{\sqrt{2}}\left(\pi - 2\ln(\sqrt{2}+1)\right). \tag{7.12}$$

This is now verified by evaluating the series in (7.11). Start by integrating the geometric series

$$\sum_{k=1}^\infty (-1)^k x^{4k-2} = \frac{x^2}{1+x^4} \tag{7.13}$$

to produce

$$\sum_{k=1}^\infty \frac{(-1)^{k-1}}{4k-1} = \int_0^1 \frac{x^2\,dx}{1+x^4}. \tag{7.14}$$

The factorization $x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$ gives the integral by the method of partial fractions.

8. An evaluation via differential equations

This section describes a method to evaluate the entries in Section **3.525** by employing differential equations.

EXAMPLE 8.1. Entry **3.525.1** states that

$$\int_0^\infty \frac{\sinh ax}{\sinh \pi x} \frac{dx}{1+x^2} = -\frac{a}{2}\cos a + \frac{1}{2}\sin a \,\ln[2(1+\cos a)]. \tag{8.1}$$

To verify this evaluation define

$$y(a) = \int_0^\infty \frac{\sinh ax}{\sinh \pi x} \frac{dx}{1+x^2}. \tag{8.2}$$

Then

$$y''(a) + y(a) = \int_0^\infty \frac{\sinh ax}{\sinh \pi x}\, dx = \frac{1}{2}\tan\frac{a}{2} \tag{8.3}$$

according to **3.511.2**. The equation (8.3) is solved by the method of variation of parameters. The general solution is of the form

$$y(a) = (u_1(a) + A)\cos a + (u_2(a) + B)\sin a \tag{8.4}$$

where the (unknown) functions u_1, u_2 are determined by solving the system

$$\begin{aligned} u_1' \cos a + u_2' \sin a &= 0 \\ -u_1' \sin a + u_2' \cos a &= \frac{1}{2}\tan\frac{a}{2}. \end{aligned}$$

The solution to this system is

$$u_1(a) = \frac{1}{2}(\sin a - a) \text{ and } u_2(a) = \frac{1}{2}(\ln(1+\cos a) - \cos a). \tag{8.5}$$

The constants A and B in (8.4) are obtained from the values $y(0) = 0$ and

$$y(\pi/2) = \int_0^\infty \frac{1}{2\cosh(\pi x/2)} \frac{dx}{1+x^2} = \frac{\ln 2}{2} \tag{8.6}$$

according to **3.522.8**. This establishes (8.1).

Differentiation of (8.1) gives **3.525.3**

$$\int_0^\infty \frac{\cosh ax}{\sinh \pi x} \frac{x\,dx}{1+x^2} = \frac{1}{2}(a\sin a - 1) + \frac{\cos a}{2}\ln[2(1+\cos a)]. \tag{8.7}$$

The same procedure gives the remaining integrals in Section **3.525**, namely **3.525.2**

$$\int_0^\infty \frac{\sinh ax}{\sinh(\pi x/2)} \frac{dx}{1+x^2} = \frac{\pi}{2}\sin a + \frac{\cos a}{2}\ln\frac{1-\sin a}{1+\sin a} \tag{8.8}$$

and its derivative **3.525.4**

$$\int_0^\infty \frac{\cosh ax}{\sinh(\pi x/2)} \frac{x\,dx}{1+x^2} = \frac{\pi}{2}\cos a - 1 - \frac{\sin a}{2}\ln\frac{1+\sin a}{1-\sin a}, \tag{8.9}$$

as well as **3.525.6**

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x} \frac{dx}{1+x^2} = 2\cos(a/2) - \frac{\pi}{2}\cos a - \sin a \,\ln\tan\frac{a+\pi}{4} \tag{8.10}$$

and its derivative **3.525.5**

$$\int_0^\infty \frac{\sinh ax}{\cosh \pi x} \frac{x\,dx}{1+x^2} = -2\sin(a/2) + \frac{\pi}{2}\sin a - \cos a \,\ln\tan\frac{a+\pi}{4}. \tag{8.11}$$

9. Squares in denominators

Section **3.527** contains a collection of integrals where the integrand is a combination of powers of the integration variable and a rational function of hyperbolic functions. The majority of them contain the square of sinh or cosh in the denominator. These integrals are evaluated in this section.

EXAMPLE 9.1. Entry **3.527.1** states that

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sinh^2(ax)} = \frac{4\Gamma(\mu)\,\zeta(\mu-1)}{(2a)^\mu}. \tag{9.1}$$

The change of variables $t = ax$ shows that it is sufficient to consider the case $a = 1$. This is

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{\sinh^2 t} = 2^{2-\mu}\Gamma(\mu)\zeta(\mu-1). \tag{9.2}$$

The integral to be evaluated is

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{\sinh^2 t} = 4\int_0^\infty \frac{t^{\mu-1}\,dt}{(e^t-e^{-t})^2} = 4\int_0^\infty \frac{t^{\mu-1}e^{-2t}\,dt}{(1-e^{-2t})^2}.$$

Expand the integrand into series to obtain

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{\sinh^2 t} = 4\sum_{n=1}^\infty n\int_0^\infty t^{\mu-1}e^{-2nt}\,dt. \tag{9.3}$$

The change of variables $v = 2nt$ yields

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{\sinh^2 t} = 4\sum_{n=1}^\infty \frac{1}{n^{\mu-1}} \times \frac{1}{2^\mu}\int_0^\infty v^{\mu-1}e^{-v}\,dv. \tag{9.4}$$

The series gives the Riemann zeta function term $\zeta(\mu-1)$ and the integral is $\Gamma(\mu)$.

The special case $\mu = 3$ gives

$$\int_0^\infty \frac{x^2\,dx}{\sinh^2 x} = \frac{1}{2}\Gamma(3)\zeta(2). \tag{9.5}$$

The values $\Gamma(3) = 2$ and $\zeta(2) = \pi^2/6$ gives the evaluation of entry **3.527.12**

$$\int_{-\infty}^\infty \frac{x^2\,dx}{\sinh^2 x} = \frac{\pi^2}{3}. \tag{9.6}$$

The identity

$$\zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!}|B_{2m}| \tag{9.7}$$

that provides the values of the Riemann zeta function at even integers in terms of the Bernoulli numbers B_{2m} gives **3.527.2** (in the scaled form $a = 1$)

$$\int_0^\infty \frac{x^{2m}\,dx}{\sinh^2 x} = \pi^{2m}|B_{2m}|. \tag{9.8}$$

EXAMPLE 9.2. Entry **3.527.3** states that

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh^2 x} = 2^{2-\mu}(1-2^{2-\mu})\Gamma(\mu)\zeta(\mu-1) \tag{9.9}$$

for $\mu \neq 2$ and

$$\int_0^\infty \frac{x\,dx}{\cosh^2 x} = \ln 2 \tag{9.10}$$

for the corresponding value for $\mu = 2$. This integral also appears as **3.527.4**. The evaluation proceeds as in the previous example to produce

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh^2 x} = -2^{2-\mu}\Gamma(\mu)\sum_{k=1}^\infty \frac{(-1)^k}{k^{\mu-1}}. \tag{9.11}$$

The last series can be expressed in terms of the Riemann zeta function by splitting the cases k even and odd to produce the identity

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^{\mu-1}} = (2^{2-\mu}-1)\zeta(\mu-1) \tag{9.12}$$

for $\mu > 1$. The case $\mu = 2$ is obtained from the elementary value

$$\sum_{k=1}^\infty \frac{(-1)^k}{k} = -\ln 2. \tag{9.13}$$

As in the previous example, the identity (9.7) gives

$$\int_0^\infty \frac{x^{2m}\,dx}{\cosh^2 x} = \frac{(2^{2m}-2)}{2^{2m}\pi^{2m}}|B_{2m}|. \tag{9.14}$$

This appears as **3.527.5**.

The same procedure provides the evaluation

$$\int_0^\infty x^{\mu-1}\frac{\cosh x\,dx}{\sinh^2 x} = 2\Gamma(\mu)\zeta(\mu-1)(1-2^{1-\mu}), \tag{9.15}$$

which appears as entry **3.527.16**. The special case $\mu = 2m+2$ appears as entry **3.527.9**

$$\int_0^\infty x^{2m+1}\frac{\cosh x}{\sinh^2 x}\,dx = \frac{2^{2m+1}-1}{2^{2m}}(2m+1)!\zeta(2m+1), \tag{9.16}$$

and $\mu = 2m+1$ provides entry **3.527.10** in the form

$$\int_0^\infty x^{2m}\frac{\cosh x}{\sinh^2 x}\,dx = (2^{2m-1}-1)\pi^{2m}|B_{2m}| \tag{9.17}$$

employing (9.7). Entry **3.527.13**

$$\int_0^\infty x^2\frac{\cosh x}{\sinh^2 x}\,dx = \frac{\pi^2}{2} \tag{9.18}$$

is the special case $\mu = 3$.

10. Two integrals giving beta function values

This final section presents the evaluation of the two integrals that constitute Section **3.512**.

EXAMPLE 10.1. Entry **3.512.1** states that

$$\int_0^\infty \frac{\cosh 2\beta x}{\cosh^{2\nu} ax}\, dx = \frac{4^{\nu-1}}{a} B\left(\nu + \frac{\beta}{a}, \nu - \frac{\beta}{a}\right). \tag{10.1}$$

The change of variables $t = ax$ and replacing β/a by c provides an equivalent form of the entry:

$$\int_0^\infty \frac{\cosh 2ct}{(\cosh t)^{2\nu}}\, dt = 4^{\nu-1} B(\nu + c, \nu - c). \tag{10.2}$$

The beta function appearing in the answer is defined by its integral representation

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{10.3}$$

To evaluate the left-hand side of (10.2), write the integrand in exponential form and let $w = e^{-ct}$ to obtain

$$\int_0^\infty \frac{e^{2(c-\nu)t} + e^{-2(c+\nu)t}}{(1+e^{-2t})^{2\nu}} = \int_0^1 \frac{w^{\nu+c} + w^{\nu-c}}{(1+w)^{2\nu}}\, dw. \tag{10.4}$$

The result now comes from the integral representation

$$B(x,y) = \int_0^1 \frac{w^{x-1} + w^{y-1}}{(1+w)^{x+y}}\, dw, \tag{10.5}$$

that appears as entry **8.380.5** of [**1**]. An elementary proof of it from (10.3) starts with the change of variables $s = t/(1-t)$ to produce

$$B(x,y) = \int_0^\infty \frac{s^{x-1}\, ds}{(1+s)^{x+y}} \tag{10.6}$$

given as entry **8.380.3** and then transform the integral to $[0,1]$ by splitting into $[0,1]$ and $[1,\infty)$ and moving the second integral to $[0,1]$ by $s_1 = 1/s$.

The special case $\beta = 0$ gives

$$\int_0^\infty \frac{dx}{(\cosh x)^{2\mu}} = 4^{\mu-1} B(\mu,\mu) \tag{10.7}$$

and letting $t = ax$ gives

$$\int_0^\infty \frac{dx}{(\cosh at)^{2\mu}} = \frac{4^{\mu-1}}{a} B(\mu,\mu) \tag{10.8}$$

Differentiate with respect to the parameter a to produce

$$\int_0^\infty \frac{x \sinh ax\, dx}{(\cosh ax)^{2\mu+1}} = \frac{2^{2\mu-2}}{\mu a^2} B(\mu,\mu). \tag{10.9}$$

The duplication formula of the gamma function

$$\Gamma(2\mu) = \frac{2^{2\mu-1}}{\sqrt{\pi}}\Gamma(\mu)\Gamma(\mu+\tfrac{1}{2}) \tag{10.10}$$

transforms (10.9) into

$$\int_0^\infty \frac{x \sinh ax \, dx}{(\cosh ax)^{2\mu+1}} = \frac{\sqrt{\pi}}{4\mu a^2} \frac{\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})}. \tag{10.11}$$

This appears as entry **3.527.11**.

Example 10.2. The last entry in Section 3.512 is **3.512.2**

$$\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x}\, dx = \frac{1}{2} B\left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2}\right). \tag{10.12}$$

Two proofs of this evaluation are given here. The first one is elementary and the second one enters the realm of hypergeometric functions.

The first proof begins with the change of variables $w = \cosh x$ to obtain

$$\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x}\, dx = \int_1^\infty (w^2-1)^{\frac{\mu-1}{2}} w^{-\nu}\, dw \tag{10.13}$$

followed by the change of variables $t = w^{-2}$ to produce

$$\frac{1}{2}\int_0^1 t^{\frac{\nu-\mu}{2}-1}(1-t)^{\frac{\nu-\mu}{2}-1}\, dt = \frac{1}{2} B\left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2}\right). \tag{10.14}$$

The second proof begins by writing the integrand as exponentials to obtain

$$\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x}\, dx = 2^{\nu-\mu-1}\int_0^1 t^{\nu/2-\mu/2-1}(1-t)^\mu(1+t)^{-\nu}\, dt$$

after the change of variable $t = e^{-2x}$. The integral representation **9.111** states that

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = B(b, c-b)\,{}_2F_1(a,b;c;z).$$

It follows that

$$\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x}\, dx = 2^{\nu-\mu-1} B\left(\frac{\nu-\mu}{2}, 1+\mu\right) {}_2F_1\left(\nu, \frac{\nu-\mu}{2}; 1+\frac{\mu+\nu}{2}; -1\right).$$

Now use **9.131.1** ${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; z/(z-1)\right)$ to transform the integral to the value of a hypergeometric function with $z = 1/2$. The quadratic transformation **9.133** ${}_2F_1\left(2a, 2b; a+b+\frac{1}{2}; z\right) = {}_2F_1\left(a, b; a+b+\frac{1}{2}; 4z(1-z)\right)$ transform it to the value of a hypergeometric function with $z = 1$. The result now follows from the evaluation

$${}_2F_1\left(a,b;c;1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

11. The last two entries of Section 3.525

This section presents a new technique that will produce evaluations of entries **3.525.7** and **3.525.8**. This completes the verification of all entries in this section that started in Section 8.

The first step is the computation of a Laplace transform.

LEMMA 11.1. *The identity*

$$\int_0^\infty \frac{e^{-st}\,dt}{\cosh\lambda t+\cos\lambda p} = \frac{2}{\sin\lambda p}\sum_{n=1}^\infty (-1)^{n-1}\frac{\sin(\lambda pn)}{s+\lambda n} \tag{11.1}$$

holds.

PROOF. The factorization

$$\cosh\lambda t+\cos\lambda p = \frac{e^{\lambda t}}{2}\left(1+e^{-2\lambda t}+e^{-\lambda t+i\lambda p}+e^{-\lambda t-i\lambda p}\right) \tag{11.2}$$

gives the decomposition

$$\begin{aligned}\frac{e^{-st}}{\cosh\lambda t+\cos\lambda p} &= \frac{2e^{-(\lambda+s)t}}{(1+e^{-\lambda(t-ip)})(1+e^{-\lambda(t+ip)})} \\ &= \frac{e^{-st}}{\sin\lambda p}\left(\frac{1}{1+e^{-\lambda(t+ip)}}-\frac{1}{1+e^{-\lambda(t-ip)}}\right) \\ &= -\frac{2e^{st}}{\sin\lambda p}\sum_{n=0}^\infty(-1)^n e^{-\lambda tn}\sin\lambda pn.\end{aligned}$$

The result now follows by integration. □

EXAMPLE 11.1. The special case $\lambda=1$ and $p=\pi-q$ in the lemma gives entry **3.543.2**:

$$\int_0^\infty \frac{e^{-st}\,dt}{\cosh t-\cos q} = \frac{2}{\sin q}\sum_{n=1}^\infty\frac{\sin(qn)}{s+n}. \tag{11.3}$$

EXAMPLE 11.2. Entry **3.511.5** is established next. Its value is employed in the next example. This entry states

$$\int_0^\infty \frac{\sinh ax\cosh bx}{\sinh cx}\,dx = \frac{\pi}{2c}\left(\frac{\sin\frac{\pi a}{c}}{\cos\frac{\pi a}{c}+\cos\frac{\pi b}{c}}\right)$$

The proof starts by expressing the integrand in exponential form to obtain

$$\int_0^\infty \frac{\sinh ax\cosh bx}{\sinh cx}\,dx = \frac{1}{2}\int_0^\infty \frac{e^{-cx}(e^{ax}-e^{-ax})(e^{bx}+e^{-bx})}{1-e^{-2cx}}\,dx$$

and use the change of variables $t=e^{-2cx}$ to produce

$$\int_0^\infty \frac{\sinh ax\cosh bx}{\sinh cx}\,dx = \frac{1}{4c}\int_0^1 \frac{t^{-A-B-1/2}+t^{-A+B-1/2}-t^{A-B-1/2}-t^{A+B-1/2}}{1-t}\,dt$$

with $A = \frac{a}{2c}$ and $B = \frac{b}{2c}$. Using the formula

$$\int_0^1 \frac{1-x^{a-1}}{1-x}\,dx = \psi(a) + \gamma$$

given as entry **3.265** (established in [**3**]), it follows that

$$\int_0^\infty \frac{\sinh ax \cosh bx}{\sinh cx}\,dx = \frac{1}{4c}\left(\psi\left(\tfrac{1}{2}+A-B\right) - \psi\left(\tfrac{1}{2}-A+B\right) + \psi\left(\tfrac{1}{2}+A+B\right) - \psi\left(\tfrac{1}{2}-A-B\right)\right).$$

The result now follows from the identity

$$\psi(\tfrac{1}{2}+z) - \psi(\tfrac{1}{2}-z) = \pi \tan \pi z.$$

EXAMPLE 11.3. Entry **3.525.7** is

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(bx)}\,\frac{s}{s^2+x^2}\,dx = \pi \sum_{n=1}^\infty \frac{\sin\left(\frac{n(b-a)}{b}\pi\right)}{bs+n\pi}. \tag{11.4}$$

The evaluation employs the Laplace transform

$$\int_0^\infty e^{-st}\cos xt\,dt = \frac{s}{s^2+x^2} \tag{11.5}$$

and entry **3.511.5** given in the previous example:

$$\begin{aligned}
\int_0^\infty \frac{\sinh(ax)}{\sinh(bx)}\frac{s}{s^2+x^2}\,dx &= \int_0^\infty e^{-st}\left\{\int_0^\infty \frac{\sinh(ax)}{\sinh(bx)}\cos xt\,dx\right\}dt \\
&= \int_0^\infty e^{-st}\left\{\frac{\pi}{2b}\,\frac{\sin\frac{\pi a}{b}}{\cosh\frac{\pi t}{b}+\cos\frac{\pi a}{b}}\right\}dt \\
&= \frac{\pi}{2b}\sin\frac{\pi a}{b}\int_0^\infty \frac{e^{-st}\,dt}{\cosh\frac{\pi t}{b}+\cos\frac{\pi a}{b}}.
\end{aligned}$$

The proof concludes by choosing $\lambda = \pi/b$ and $a = p$ in Lemma 11.1 and using $\sin\left(n(b-a)\pi/b\right) = (-1)^{n-1}\sin\left(n\pi a/b\right)$.

EXAMPLE 11.4. Differentiation entry **3.525.7** with respect to the parameter a gives

$$\int_0^\infty \frac{\cosh(ax)}{\sinh(bx)}\,\frac{x}{s^2+x^2}\,dx = \frac{\pi}{bs}\sum_{n=1}^\infty \frac{(-1)^{n-1}\pi n}{bs+\pi n}\cos\frac{\pi a n}{b}. \tag{11.6}$$

To simplify this expression use $\dfrac{\pi n}{bs+\pi n} = 1 - \dfrac{bs}{bs+\pi n}$ and split the series using the Fourier expansion

$$\sum_{n=1}^\infty (-1)^{n-1}\cos\frac{\pi a n}{b} = \frac{1}{2}. \tag{11.7}$$

This final result is entry **3.525.8**

$$\int_0^\infty \frac{\cosh ax}{\sinh bx} \frac{x\,dx}{s^2+x^2} = \frac{\pi}{2bs} + \pi \sum_{n=1}^\infty \frac{\cos \frac{n(b-a)}{b}\pi}{bs+n\pi}. \tag{11.8}$$

The series in (11.6) and (11.7) are both Abel-convergent. The reader is invited to verify that the series (11.8) is convergent and reduces to (8.7) when $b=\pi$ and $s=1$.

Remark 11.1. Section 4.11 of [**1**] contain many analogous formulas as those considered here. For instance, entry **3.525.1**

$$\int_0^\infty \frac{\sinh ax}{\sinh \pi x} \frac{dx}{1+x^2} = -\frac{a}{2}\cos a + \frac{1}{2}\sin a \, \ln[2(1+\cos a)] \tag{11.9}$$

is related to entry **4.113.3**

$$\int_0^\infty \frac{\sin ax}{\sinh \pi x} \frac{dx}{1+x^2} = -\frac{a}{2}\cosh a + \frac{1}{2}\sinh a \, \ln[2(1+\cosh a)]. \tag{11.10}$$

The right-hand side of the last entry appears in [**1**] in the equivalent form

$$-\frac{a}{2}\cosh a + \sinh a \, \ln[2\cosh a/2]. \tag{11.11}$$

A systematic study of this correspondance and the evaluation of the integrals appearing in Section 4.11 will be presented in a future publication.

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