

## The integrals in Gradshteyn and Ryzhik. Part 5: Some trigonometric integrals

Tewodros Amdeberhan, Luis A. Medina, and Victor H. Moll

ABSTRACT. We present evaluations and provide proofs of definite integrals involving the function  $x^p \cos^n x$ . These formulae are generalizations of 3.761.11 and 3.822.1, among others, in the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik.

### 1. Introduction

The table of integrals [4] contains a large variety of evaluations of the type

$$(1.1) \quad I = \int_a^b A(x)R(\sin x, \cos x) dx$$

where  $A$  is an algebraic function,  $R$  is rational and  $-\infty \leq a < b \leq \infty$ . We present a systematic discussion of two families of integrals of this type. This paper is part of a general program started in [9, 10, 11, 12] intended to provide proofs and context to the formulas in [4].

The first class considered here corresponds to the complete integrals

$$(1.2) \quad c(n, p) := \int_0^{\pi/2} x^p \cos^n x dx,$$

and

$$(1.3) \quad s(n, p) := \int_0^{\pi/2} x^p \sin^n x dx,$$

where  $n, p \in \mathbb{N}$ . In section 2 we present closed-form expressions for these integrals. These expressions involve the sums

$$(1.4) \quad \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \dots k_j^2},$$

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that are closely related to the multiple zeta values

$$(1.5) \quad \zeta(i_1, i_2, \dots, i_s) = \sum_{0 < k_1 < k_2 < \dots < k_s} \frac{1}{k_1^{i_1} k_2^{i_2} \dots k_s^{i_s}}.$$

The reader will find in Section 3.4 of [3] an introduction to these sums.

In general, one does not expect such elementary evaluations to extend to  $p \notin \mathbb{N}$ . For example, the change of variables  $x = \pi t^2/2$  produces

$$(1.6) \quad \int_0^{\pi/2} x^{-1/2} \cos x \, dx = \sqrt{2\pi} \int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt.$$

The latter integral is evaluated in terms of the *cosine Fresnel* function

$$(1.7) \quad \text{FresnelC}[x] := \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt,$$

which indeed is not an elementary function.

The second class considered here presents generalizations of the formula **3.822.2** in [4] stated as

$$(1.8) \quad \int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}} dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \quad n \in \mathbb{N}.$$

The integral in (1.8) can be transformed via  $t = x^2$  to provide the evaluation of

$$(1.9) \quad \int_0^\infty \cos^{2n+1} t^2 dt,$$

that is given as the case  $p = 2$  in Theorem 3.2.

Section 3 contains analytic expressions for the generalizations

$$(1.10) \quad C_n(p, b) := \int_0^\infty x^{-p} \cos^{2n+1}(x+b) dx,$$

and

$$(1.11) \quad S_n(p, b) := \int_0^\infty x^{-p} \sin^{2n+1}(x+b) dx.$$

The last section also contains some evaluations obtained by differentiation with respect to parameters. An illustrative example is

$$(1.12) \quad \int_0^\infty \int_0^\infty \frac{\log x \log y}{\sqrt{xy}} \cos(x+y) dx dy = (\gamma + 2 \log 2) \pi^2,$$

that is equivalent to

$$(1.13) \quad \int_0^\infty \int_0^\infty \log x \log y \cos(x^2 + y^2) dx dy = \frac{1}{16} (\gamma + 2 \log 2) \pi^2.$$

A generalization of this evaluation appears as Example 3.3.

The method described in the present work gives impetus to a class of integrals that are closely related to the particular integral computations addressed in this paper.

## 2. The first example

In this section we present the evaluation in closed-form of the definite integrals

$$(2.1) \quad c(n, p) := \int_0^{\pi/2} x^p \cos^n x \, dx.$$

A special case of this appears as **3.822.1** in [4].

The first step towards the evaluation of  $c(n, p)$  is to produce a recurrence.

**Theorem 2.1.** The integral  $c(n, p)$  satisfies the recurrence

$$(2.2) \quad c(n, p) = \frac{n-1}{n} c(n-2, p) - \frac{p(p-1)}{n^2} c(n, p-2),$$

for  $n \geq 2, p \geq 2$ .

PROOF. The identity  $\cos^2 x = 1 - \sin^2 x$  yields

$$(2.3) \quad c(n, p) = c(n-2, p) - I(n, p)$$

where

$$I(n, p) := \int_0^{\pi/2} x^p \cos^{n-2} x \sin^2 x \, dx.$$

Now

$$\begin{aligned} I(n, p) &= \int_0^{\pi/2} x^p \sin x \times \frac{d}{dx} \left( -\frac{1}{n-1} \cos^{n-1} x \right) dx \\ &= \frac{1}{2n-1} \int_0^{\pi/2} (x^p \cos x + px^{p-1} \sin x) \cos^{n-1} x \, dx \\ &= \frac{c(n, p)}{n-1} + \frac{p}{n-1} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \, dx. \end{aligned}$$

Moreover

$$\begin{aligned} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \, dx &= \int_0^{\pi/2} x^{p-1} \frac{d}{dx} \left( -\frac{1}{n} \cos^n x \right) dx \\ &= \frac{p-1}{n} c(n, p-2). \end{aligned}$$

□

**Strategy:** According to (2.2), the integral  $c(n, p)$  can be evaluated in terms of the initial values given in the table. The indices  $m$  and  $q$  have the same parity as  $n$  and  $p$  respectively and range over  $0 \leq m \leq n$  and  $0 \leq q \leq p$ .

$n$ modulo 2	$p$ modulo 2	initial conditions
0	0	$c(m, 0) \quad c(0, q)$
1	0	$c(m, 0) \quad c(1, q)$
0	1	$c(m, 1) \quad c(0, q)$
1	1	$c(m, 1) \quad c(1, q)$

We now evaluate the initial conditions  $c(n, 0)$ ,  $c(n, 1)$ ,  $c(0, p)$  and  $c(1, p)$ .

**The expression for  $c(0, p)$ .**

The computation of the identity

$$(2.4) \quad c(0, p) = \frac{1}{p+1} \left(\frac{\pi}{2}\right)^{p+1}$$

is immediate.

**The expression for  $c(n, 0)$ .**

This is classical. The result appears as **3.621.3** and **3.621.4** in [4].

**Theorem 2.2. (Wallis' formula and companion).** Let  $n \in \mathbb{N}_0$ . Then

$$(2.5) \quad c(2n, 0) = \frac{\pi}{2^{2n+1}} \binom{2n}{n},$$

and

$$(2.6) \quad c(2n+1, 0) = \frac{2^{2n}}{(2n+1) \binom{2n}{n}}.$$

The shortest proof of Theorem 2.2 employs the representation

$$(2.7) \quad c(n, 0) = \int_0^{\pi/2} \cos^n x \, dx = 2^{n-1} B\left(\frac{n+1}{2}, \frac{n+1}{2}\right),$$

that appears as **3.621.1** in [4]. Here  $B$  is the *Euler's beta function* defined by the integral

$$(2.8) \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

The expression (2.7) follows from the change of variables  $t = \cos u$ . To express (2.5) and (2.6), in terms of the beta function, employ the standard relation

$$(2.9) \quad B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)},$$

and the special values

$$(2.10) \quad \Gamma(n) = (n-1)! \quad \text{and} \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi} (2n)!}{2^{2n} n!}$$

that are valid for  $n \in \mathbb{N}$ .

The identity in Theorem 2.2, in the case  $n$  is even, that is,

$$(2.11) \quad c(2n, 0) = \int_0^{\pi/2} \cos^{2n} \theta \, d\theta = \frac{\pi}{2^{2n+1}} \binom{2n}{n},$$

is Wallis's formula and sometimes found in calculus books (see e.g. [6], page 492). To prove it, first write  $\cos^2 \theta = 1 - \sin^2 \theta$  and use integration by parts to obtain the recursion

$$(2.12) \quad c(2n, 0) = \frac{2n-1}{2n} c(2n-2, 0).$$

Then verify that the right side of (2.11) satisfies the same recurrence together with the initial value  $\pi/2$  for  $n = 0$ .

We now present a new proof of Wallis's formula (2.11) in the context of rational integrals. Extensions of the ideas in this proof have produced *rational Landen transformations*. The reader will find in [1, 2, 5, 7, 8] details on these transformations.

Start with

$$c(2n, 0) = \int_0^{\pi/2} \cos^{2n} \theta \, d\theta = \int_0^{\pi/2} \left( \frac{1 + \cos 2\theta}{2} \right)^n \, d\theta.$$

Now introduce  $\psi = 2\theta$  and expand and simplify the result by observing that the odd powers of cosine integrate to zero. The inductive proof of (2.11) requires

$$(2.13) \quad c(2n, 0) = 2^{-n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} c(2i, 0).$$

Note that  $c(2n, 0)$  is uniquely determined by (2.13) along with the initial value  $c(0, 0) = \pi/2$ . Thus (2.11) now follows from the identity

$$(2.14) \quad f(n) := \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{-2i} \binom{n}{2i} \binom{2i}{i} = 2^{-n} \binom{2n}{n}.$$

We now provide a mechanical proof of (2.14) using the theory developed by Wilf and Zeilberger, which is explained in [13, 14]; the sum in (2.14) is the example used in [14] (page 113) to illustrate their method. The command

$$ct(\text{binomial}(n, 2i) \text{binomial}(2i, i) 2^{-2i}, 1, i, n, N)$$

produces

$$(2.15) \quad f(n+1) = \frac{2n+1}{n+1} f(n),$$

and one checks that  $2^{-n} \binom{2n}{n}$  satisfies this recursion. Note that (2.12) and (2.15) are equivalent under

$$c(2n, 0) = \frac{\pi}{2^{n+1}} f(n).$$

The proof is complete.

**Closed form expression for  $c(1, p)$ .**

We now consider the evaluation of

$$(2.16) \quad c(1, p) := \int_0^{\pi/2} x^p \cos x \, dx.$$

The following evaluation appears as **3.761.11** in [4].

**Theorem 2.3.** Let  $p \in \mathbb{N}$  and  $\delta_{\text{odd}, p}$  be Kronecker's delta function at the odd integers. Then

$$(2.17) \quad c(1, p) = \sum_{k=0}^{\xi_p} (-1)^k \frac{p!}{(p-2k)!} \left( \frac{\pi}{2} \right)^{p-2k} - (-1)^{\xi_p} \delta_{\text{odd}, p} p!$$

where  $\xi_p = \lfloor \frac{p}{2} \rfloor$ .

PROOF. Both sides of the equation (2.17) satisfy the initial value problem

$$(2.18) \quad u_p - p(p-1)u_{p-2} = \left(\frac{\pi}{2}\right)^p \quad \text{and} \quad u_0 = 1, \quad u_1 = \frac{\pi-2}{2}.$$

Actually the recurrence (2.18) is obtained using integration by parts in (2.16). Iterating this recurrence yields the right hand side of (2.17).  $\square$

**Note 2.4.** The result in Theorem 2.3 can be expressed in terms of the Taylor polynomial for  $\cos x$ :

$$(2.19) \quad f_p(x) = (-1)^{\xi_p} p! \left( -1 + \sum_{k=0}^{\xi_p} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right).$$

The formula (2.17) can be restated

$$(2.20) \quad c(1, p) = \begin{cases} f_p(\pi/2), & \text{for } p \text{ odd,} \\ f'_p(\pi/2), & \text{for } p \text{ even.} \end{cases}$$

**Closed form expression for  $c(n, 1)$ :** in fact, this would be the last initial condition we require to execute the startegy outlined at the beginning of this section.

**Theorem 2.5.** The integral  $c(n, 1)$  satisfies the recurrence

$$(2.21) \quad c(n, 1) = \frac{n-1}{n} c(n-2, 1) - \frac{1}{n^2}.$$

PROOF. The identity  $\cos^2 x = 1 - \sin^2 x$  yields

$$(2.22) \quad c(n, 1) = c(n-2, 1) - J,$$

where

$$(2.23) \quad J = \int_0^{\pi/2} x \sin^2 x \cos^{n-2} x dx.$$

Integration by parts leads to

$$(2.24) \quad J = \frac{1}{n-1} \int_0^{\pi/2} (\sin x + x \cos x) \cos^{n-1} x dx.$$

This produces (2.21).  $\square$

The solution of (2.21) yields a closed-form formula for  $c(n, 1)$ .

**Theorem 2.6.** The integral  $c(n, 1)$  is given according to the parity of  $n$ , by

$$(2.25) \quad c(2n, 1) = \frac{\binom{2n}{n}}{2^{2n+2}} \left( \frac{\pi^2}{2} - \sum_{k=1}^n \frac{2^{2k}}{k^2 \binom{2k}{k}} \right),$$

for even indices. For odd indices, we have

$$(2.26) \quad c(2n+1, 1) = \frac{2^{2n}}{(2n+1) \binom{2n}{n}} \left( \frac{\pi}{2} - \sum_{k=0}^n \frac{\binom{2k}{k}}{2^{2k} (2k+1)} \right).$$

To establish this result we solve a more general recurrence than (2.21).

**Lemma 2.7.** Let  $a_n, b_n$  and  $r_n$  be sequences with  $a_n, b_n \neq 0$ . Assume that  $z_n$  satisfies

$$(2.27) \quad a_n z_n = b_n z_{n-1} + r_n, \quad n \geq 1$$

with initial condition  $z_0$ . Then

$$(2.28) \quad z_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \left( z_0 + \sum_{k=1}^n \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} r_k \right).$$

PROOF. Introduce the integrating factor  $d_n$  with the property that  $d_n b_n = d_{n-1} a_{n-1}$ . The recurrence (2.27) becomes

$$(2.29) \quad D_n - D_{n-1} = d_n r_n,$$

where  $D_n = d_n a_n z_n$ . Therefore, by telescoping,

$$(2.30) \quad D_n = D_0 + \sum_{k=1}^n d_k r_k,$$

with  $D_0 = d_0 a_0 z_0$ . To find the integrating factor, observe that

$$(2.31) \quad \frac{d_n}{d_{n-1}} = \frac{a_{n-1}}{b_n}.$$

Thus

$$(2.32) \quad d_n = \frac{a_0 a_1 \cdots a_{n-1}}{b_1 b_2 \cdots b_n} d_0.$$

Replacing in (2.30) yields (2.28). □

**Corollary 2.8.** Let  $n \in \mathbb{N}$  and assume that  $z_n$  satisfies

$$(2.33) \quad 2n z_n = (2n-1) z_{n-1} + r_n, \quad n \geq 1,$$

with the initial condition  $z_0$ . Let  $\lambda_j = 2^{2j} \binom{2j}{j}^{-1}$ , then

$$(2.34) \quad z_n = \frac{1}{\lambda_n} \left( z_0 + \sum_{k=1}^n \frac{\lambda_k r_k}{2k} \right).$$

PROOF. Use  $a_n = 2n$  and  $b_n = 2n-1$  in Lemma 2.7. □

We now apply Lemma 2.7 on the recurrence (2.21), repeated here for convenience to the reader,

$$c(n, 1) = \frac{n-1}{n} c(n-2, 1) - \frac{1}{n^2}.$$

Observe that this recurrence splits naturally into even and odd branches. The value of  $c(2n, 1)$  is determined completely by  $c(0, 1)$ , and  $c(2n+1, 1)$  by  $c(1, 1)$ . Hence, there is no computational interaction between  $c(2n, 1)$  and  $c(2n+1, 1)$ . Let  $x_n = c(2n, 1)$  so that  $x_n$  satisfies

$$(2.35) \quad 2n x_n = (2n-1) x_{n-1} - \frac{1}{4n},$$

with the initial condition

$$(2.36) \quad x_0 = c(0, 1) = \frac{\pi^2}{8}.$$

Similarly,  $y_n = c(2n + 1, 1)$ , the odd component of  $c(n, 1)$ , satisfies

$$(2.37) \quad (2n + 1)y_n = 2ny_{n-1} - \frac{1}{2n + 1}$$

and the initial condition

$$(2.38) \quad y_0 = c(1, 1) = \frac{\pi}{2} - 1.$$

The expressions for  $z_n$  in Lemma 2.7 yield the formulas for  $c(2n, 1)$  and also  $c(2n + 1, 1)$  in Theorem 2.6. The proof is complete.

**Note 2.9.** The finite sums in (2.25) and (2.26) do not have closed-form, but it is a classical result that, in the limit,

$$(2.39) \quad \sum_{k=1}^{\infty} \frac{2^{2k}}{k^2 \binom{2k}{k}} = \frac{\pi^2}{2}$$

and

$$(2.40) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k} (2k + 1)} = \frac{\pi}{2}.$$

**Note 2.10.** Formula 3.821.3 in [4] gives formulas equivalent to (2.25) and (2.26), respectively.

Finally, we conclude this section by presenting the sought for closed form expression for the integral  $c(n, p)$ , for arbitrary  $n, p \in \mathbb{N}$ . The recurrence (2.2), in the case of even indices  $n$ , becomes

$$(2.41) \quad 2nX_n(p) = (2n - 1)X_{n-1}(p) - \frac{p(p-1)}{2n}X_n(p-2)$$

where  $X_n(p) = c(2n, p)$ . The initial value

$$(2.42) \quad X_0(p) = \frac{1}{(p+1)2^{p+1}}\pi^{p+1}$$

given in (2.4) and the recurrence (2.41) show the existence of rational numbers  $a_{n,p,p+1-2j}$  such that

$$(2.43) \quad X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j},$$



with  $\xi_p = \lfloor \frac{p}{2} \rfloor$ . The recurrence (2.41) is now expanded as

$$(2.44) \quad 2n \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j} = (2n-1) \sum_{j=0}^{\xi_p} a_{n-1,p,p+1-2j} \pi^{p+1-2j} \\ - \frac{p(p-1)}{2n} \sum_{j=0}^{\xi_{p-1}} a_{n,p-2,p-1-2j} \pi^{p-1-2j}.$$

The fact that the coefficients  $a_{n,p,j} \in \mathbb{Q}$  allows us to match the corresponding powers of  $\pi$  in (2.44). The highest order term is  $\pi^{p+1}$ . Only two of the sums contain this power, therefore

$$(2.45) \quad 2na_{n,p,p+1} = (2n-1)a_{n-1,p,p+1}.$$

The initial condition

$$(2.46) \quad a_{0,p,p+1} = \frac{1}{(p+1)2^{p+1}}$$

comes from (2.42). The solution to the initial value problem (2.45, 2.46) is then found using Corollary 2.8 (here  $r_n = 0$ ), namely that

$$(2.47) \quad a_{n,p,p+1} = \frac{\binom{2n}{n}}{(p+1)2^{2n+p+1}}.$$

The coefficient of the next highest power  $\pi^{p-1}$ , in (2.44), yields the recurrence

$$(2.48) \quad 2na_{n,p,p-1} = (2n-1)a_{n-1,p,p-1} - \frac{p(p-1)}{2n} a_{n,p-2,p-1}.$$

Observe that the last term in this relation is given by (2.47). Moreover, (2.42) shows that  $a_{0,p,p-1} = 0$ . The solution to (2.48), following Corollary 2.8, is

$$(2.49) \quad a_{n,p,p-1} = -\frac{p \binom{2n}{n}}{2^{2n+p+1}} \sum_{k_1=1}^n \frac{1}{k_1^2}.$$

The next power of  $\pi$  in (2.44) produces

$$(2.50) \quad 2na_{n,p,p-3} = (2n-1)a_{n-1,p,p-3} + \frac{p(p-1)(p-2)}{n2^{2n+p}} \binom{2n}{n} \sum_{k_1=1}^n \frac{1}{k_1^2},$$

with  $a_{0,p,p-3} = 0$ . One more use of Corollary 2.8 yields

$$(2.51) \quad a_{n,p,p-3} = \frac{\binom{2n}{n} p!}{2^{2n+p+1} (p-3)!} \sum_{k_2=1}^n \sum_{k_1=1}^{k_2} \frac{1}{k_1^2 k_2^2}.$$

This procedure can be repeated until all descending powers of  $\pi$  have been exhausted, hence a complete closed form for the integrals  $c(n, p)$  will be made possible.

**Theorem 2.11.** Let  $n, p \in \mathbb{N}$  and let  $\xi_p = \lfloor \frac{p}{2} \rfloor$ . Then the even branches  $X_n(p) = c(2n, p)$  of the integral

$$(2.52) \quad c(n, p) = \int_0^{\pi/2} x^p \cos^n x \, dx$$

are given by

$$(2.53) \quad X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j} + \delta_{\text{odd},p} \cdot a_{n,p}^*,$$

and the value of  $a_{n,p,p+1-2j}$  for  $p \geq 2$  and  $0 \leq j \leq \xi_p$  is given by

$$a_{n,p,p+1-2j} = \frac{(-1)^j \binom{2n}{n} p!}{2^{2n+p+1} (p+1-2j)!} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \dots k_j^2},$$

and

$$(2.54) \quad a_{n,p}^* = \frac{(-1)^{\xi_p} \binom{2n}{n} p!}{2^{2n}} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_p \leq n} \frac{1}{k_1^2 k_2^2 \dots k_p^2} \sum_{j=1}^{k_p} \frac{2^{2j}}{j^2 \binom{2j}{j}}.$$

Similarly, for the odd branches  $Y(n, p) = c(2n+1, p)$  we have

$$(2.55) \quad Y_n(p) = \sum_{j=0}^{\xi_p} b_{n,p,p-2j} \pi^{p-2j} + \delta_{\text{odd},p} \cdot b_{n,p}^*,$$

with

$$b_{n,p,p-2j} = \frac{(-1)^j p! 2^{2n+2j-p}}{(2n+1) \binom{2n}{n} (p-2j)!} \sum_{0 \leq k_1 \leq k_2 \leq \dots \leq k_j \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \dots (2k_j+1)^2},$$

and

$$b_{n,p}^* = \frac{(-1)^{\xi_p} p! 2^{2n}}{(2n+1) \binom{2n}{n}} \sum_{0 \leq k_1 \leq k_2 \leq \dots \leq k_p \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \dots (2k_p+1)^2} \sum_{j=0}^{k_p} \frac{\binom{2j}{j}}{2^{2j} (2j+1)}.$$

### 3. Some examples on the halfline

In this section we provide an analytic expression for

$$(3.1) \quad C_n(p, b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b) \, dx,$$

and

$$(3.2) \quad S_n(p, b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b) \, dx.$$

In the table [4] the evaluation of the special case  $p = \frac{1}{2}$  and  $b = 0$ :

$$(3.3) \quad \int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}} \, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}},$$

and

$$(3.4) \quad \int_0^\infty \frac{\sin^{2n+1} x}{\sqrt{x}} dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}},$$

as **3.822.2** and **3.821.14**.

**Theorem 3.1.** Let  $0 < p < 1$  and  $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Then

$$(3.5) \quad \int_0^\infty x^{-p} \cos^{2n+1} x dx = \frac{\Gamma(1-p)}{2^{2n}} \sin\left(\frac{\pi p}{2}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}},$$

and

$$(3.6) \quad \int_0^\infty x^{-p} \sin^{2n+1} x dx = \frac{\Gamma(1-p)}{2^{2n}} \cos\left(\frac{\pi p}{2}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}.$$

PROOF. The identity  $2 \cos x = e^{ix} + e^{-ix}$  and the binomial theorem yield

$$(3.7) \quad \int_0^\infty x^{-p} \cos^{2n+1} x dx = 2^{-2n-1} \sum_{k=0}^n \binom{2n+1}{k} \int_0^\infty x^{-p} \left( e^{i(2n+1-2k)x} + e^{-i(2n+1-2k)x} \right) dx.$$

Recall the Heaviside step function defined by  $H(x) = 1$ , if  $x > 0$  and  $H(x) = 0$  otherwise. Then, each of the integrals in (3.7) is evaluated using the Fourier transform

$$(3.8) \quad \int_{-\infty}^\infty H(x) x^{-p} e^{-i\omega x} dx = \frac{\Gamma(1-p)}{|\omega|^{1-p}} \exp(-i\pi(1-p)\text{sign}(\omega)/2).$$

□

**Corollary 3.2.** Let  $p > 1$  be real and  $n \in \mathbb{N}_0$ . Then

$$(3.9) \quad \int_0^\infty \cos^{2n+1} x^p dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \cos\left(\frac{\pi}{2p}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}},$$

and

$$(3.10) \quad \int_0^\infty \sin^{2n+1} x^p dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \sin\left(\frac{\pi}{2p}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}.$$

PROOF. The change of variables  $x \mapsto x^{1/(1-p)}$  in the results of Theorem 3.1 gives the result. □

The last result described here is a further generalization of Theorem 3.1.

**Theorem 3.3.** Assume  $b \in \mathbb{R}$ ,  $0 < p < 1$  and  $n \in \mathbb{N}_0$ . Define

$$(3.11) \quad C_n(p, b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b) dx$$

and

$$(3.12) \quad S_n(p, b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b) dx.$$

Then

$$(3.13) \quad C_n(p, b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\sin(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}},$$

and

$$(3.14) \quad S_n(p, b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} \frac{\cos(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}}.$$

PROOF. Denote the left-hand side of (3.13) and (3.14) by  $f_n(b)$  and  $g_n(b)$  respectively. Differentiation with respect to the parameter  $b$  yields

$$(3.15) \quad \begin{aligned} \frac{\partial g_n}{\partial b} - (-1)^n (2n+1) f_n &= (2n+1) \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} f_j(b) \\ \frac{\partial f_n}{\partial b} + (-1)^n (2n+1) g_n &= -(2n+1) \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} g_j(b). \end{aligned}$$

Considering  $b$  and  $p$  fixed, we now show that the right-hand side of (3.13) and (3.14) satisfy the system (3.15) with the same initial conditions. This will establish the result.

In the case of the right-hand side of (3.13), it is required to check the identity

$$\begin{aligned} &2^{-2n} \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} \frac{\sin(\frac{\pi}{2}p - (2k+1)b)}{(2k+1)^{1-p}} = \\ &(2n+1) \sum_{j=0}^n (-1)^j \binom{n}{j} 2^{-2j} \sum_{k=0}^j \binom{2j+1}{j-k} \frac{\sin(\frac{\pi}{2}p - (2k+1)b)}{(2j+1)^{1-p}}. \end{aligned}$$

To verify this we compare the coefficients of the transcendental terms

$$\frac{\sin(\frac{\pi}{2}p - (2k+1)b)}{(2k+1)^{1-p}}.$$

It turns out that this question is equivalent to validating the identity

$$(3.16) \quad (-1)^k 2^{-2n} \binom{2n+1}{n-k} (2k+1) = (2n+1) \sum_{j=k}^n (-1)^j 2^{-2j} \binom{n}{j} \binom{2j+1}{j-k}$$

To this end, we employ the WZ-technology as explained in [14]. This method produces the recurrence

$$(3.17) \quad 2(n+k+1)(n+1-k)u(n+1-k) - (n+1)(2n+3)u(n, k) = 0.$$

To prove (3.16) simply check that both sides of (3.16) satisfy the recurrence (3.17) as well as the initial condition  $u(0, 0) = 1$ .

The identities

$$(3.18) \quad \begin{aligned} \int_0^\infty x^{-p} \cos(x+b) dx &= -\Gamma(1-p) \sin(b - \frac{p\pi}{2}) \\ \int_0^\infty x^{-p} \sin(x+b) dx &= \Gamma(1-p) \cos(b - \frac{p\pi}{2}), \end{aligned}$$

which are special cases of

$$(3.19) \quad \begin{aligned} \int_0^\infty x^{-p} \cos(ax + b) dx &= -a^{p-1} \Gamma(1-p) \sin(b - \frac{p\pi}{2}) \\ \int_0^\infty x^{-p} \sin(ax + b) dx &= a^{p-1} \Gamma(1-p) \cos(b - \frac{p\pi}{2}), \end{aligned}$$

show that the corresponding initial values in (3.13) (respectively 3.14) match. The evaluations (3.19) appear as **3.764.1** and **3.764.2** in [4]. To establish (3.18) expand  $\cos(x + b)$  as  $\cos x \cos b - \sin x \sin b$ , use the change of variables  $x \mapsto x^p$ , and Theorem 3.1.  $\square$

We now discuss some definite integrals that follow from Theorem 3.3.

EXAMPLE 3.1. Differentiating (3.13) with respect to  $p$  and setting  $p = \frac{1}{2}$  and  $b = 0$  gives, after the change of variables  $x \mapsto x^2$ ,

$$(3.20) \quad \begin{aligned} \int_0^\infty \log x \cos^{2n+1} x^2 dx &= -\frac{\sqrt{\pi}}{2^{2n+3}} (\pi + 2\gamma + 4 \log 2) \sum_{k=0}^n \binom{2n+1}{n-k} \frac{1}{\sqrt{4k+2}} \\ &\quad - \frac{\sqrt{\pi}}{2^{2n+2}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\log(2k+1)}{\sqrt{4k+2}}, \end{aligned}$$

where we have used the value  $\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2 \log 2)$ .

EXAMPLE 3.2. Assume  $0 < p, q < 1$ . Multiplying (3.13) by  $b^{-q}$  and integrating over the half-line yields (after replacing  $b$  by  $y$ )

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{\cos^{2n+1}(x+y)}{x^p y^q} dA &= -\Gamma(1-p)\Gamma(1-q) \cos\left(\frac{\pi(p+q)}{2}\right) \\ &\quad \times \sum_{k=0}^n \binom{2n+1}{n-k} \frac{(2k+1)^{p+q-2}}{2^{2n}}. \end{aligned}$$

In particular, for  $n = 0$ ,

$$(3.21) \quad \int_0^\infty \int_0^\infty \frac{\cos(x+y)}{x^p y^q} dA = -\Gamma(1-p)\Gamma(1-q) \cos\left(\frac{\pi(p+q)}{2}\right).$$

The derivative  $\frac{\partial^2}{\partial p \partial q}$  at  $p = q = \frac{1}{2}$  produces the evaluation

$$(3.22) \quad \int_0^\infty \int_0^\infty \frac{\log x \log y}{\sqrt{xy}} \cos(x+y) dx dy = (\gamma + 2 \log 2) \pi^2$$

that we promised in the Introduction.

EXAMPLE 3.3. Iterating the method described in the previous example yields

$$\int_{\mathbb{R}_+^n} (\cos \|x\|^2) \cdot \prod_{j=1}^n \log x_j dV = \frac{(-1)^{\Delta_n} \pi^{n/2}}{2^{2n}} \begin{cases} \operatorname{Re} \psi_n & \text{if } n \text{ is even,} \\ \operatorname{Im} \psi_n & \text{if } n \text{ is odd,} \end{cases}$$

with

$$(3.23) \quad \Delta_n = \frac{n(n+1)}{2}, \psi_n = \left( \gamma + 2 \log 2 + \frac{\pi i}{2} \right)^n e^{\pi i n / 4}.$$

Here  $\|x\|^2 = x_1^2 + \cdots + x_n^2$  and  $\gamma$  is Euler's constant. For instance, for  $n = 3$  we have

$$\int_0^\infty \int_0^\infty \int_0^\infty \log x \log y \log z \cos(x^2 + y^2 + z^2) dx dy dz = \frac{\pi^{3/2}}{8} (-16\xi^3 + 12\xi^2\pi + 6\xi\pi^2 - \pi^3),$$

where  $\xi = \gamma + 2 \log 2$ .

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DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* tamdeberhan@math.tulane.edu

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* lmedina@math.tulane.edu

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* vhm@math.tulane.edu

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DEPARTAMENTO DE MATEMÁTICA  
UNIVERSIDAD TÉCNICA FEDERICO SANTA MARÍA  
CASILLA 110-V,  
VALPARAÍSO, CHILE