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The integrals in Gradshteyn and Ryzhik. Part 8: Combinations of powers, exponentials and logarithms.

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Abstract. We describe some examples of integrals from the table of Gradshteyn and Ryzhik where the integrand is a combination of powers, exponentials and logarithms. The expressions for some of these integrals involve the Stirling numbers of the first kind.

1. Introduction

The uninitiated reader of the table of integrals by I. S. Gradshteyn and I. M. Ryzhik [**4**] will surely be puzzled by choice of integrands. In this note we provide an elementary proof of the evaluation **4.353.3**

$$\int_0^1 (ax+n+1)x^n e^{ax} \ln x \, dx = e^a \sum_{k=0}^{n} (-1)^{k-1} \frac{n!}{(n-k)!a^{k+1}} + (-1)^n \frac{n!}{a^{n+1}}. \tag{1.1}$$

We also consider the integrals

$$q_n := \int_0^1 x^n e^{-x} \ln x \, dx \tag{1.2}$$

and the companion family

$$p_n := \int_0^1 x^n e^{-x} \, dx. \tag{1.3}$$

The integral q_n corresponds to the case $a = -1$ in (1.1). Section 3 provides closed-form expressions for p_n and q_n. Section 4 considers the generalization

$$P_n(a) = \int_0^1 x^n e^{-ax} \, dx \text{ and } Q_n(a) = \int_0^1 x^n e^{-ax} \ln x \, dx. \tag{1.4}$$

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The main result of this section is the closed-form expressions

$$P_n(a) := \int_0^1 x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}\left(1 - e^{-a}\sum_{k=0}^{n}\frac{a^k}{k!}\right), \tag{1.5}$$

and

$$Q_n(a) := \int_0^1 x^n e^{-ax}\ln x\,dx = \frac{n!}{a^{n+1}}\left[\sum_{k=1}^{n}\frac{1}{k}\left(1 - e^{-a}\sum_{j=0}^{k-1}\frac{a^j}{j!}\right) + aQ_0(a)\right],$$

where

$$Q_0(a) = \int_0^1 e^{-ax}\ln x\,dx = -\frac{1}{a}\left(\gamma + \ln a + \Gamma(0,a)\right), \tag{1.6}$$

and $\Gamma(0,a)$ is the incomplete gamma function defined by

$$\Gamma(a,x) := \int_x^\infty t^{a-1}e^{-t}\,dt. \tag{1.7}$$

2. The evaluation of 4.353.3

The identity

$$\frac{d}{dx}\left(x^{n+1}e^{ax}\right) = (ax+n+1)x^n e^{ax} \tag{2.1}$$

and integration by parts yield

$$\int_0^1 (ax+n+1)x^n e^{ax}\ln x\,dx = -\int_0^1 x^n e^{ax}\,dx. \tag{2.2}$$

This last integral appears as **3.351.1** in [**4**]. We have obtained a closed-form expression for it in [**2**]. A new proof is presented in Section 4.

A closed form expression for the right hand side of (2.2) is obtained from

$$\int_0^1 x^n e^{ax}\,dx = \left(\frac{d}{da}\right)^n \frac{e^a-1}{a}. \tag{2.3}$$

The symbolic evaluation of (2.3) for small values of $n \in \mathbb{N}$ suggests the existence of a polynomial $P_n(a)$ such that

$$\int_0^1 x^n e^{ax}\,dx = \frac{(-1)^{n+1}\,n!}{a^{n+1}} + \frac{P_n(a)}{a^{n+1}}e^a. \tag{2.4}$$

The next lemma confirms the existence of this polynomial.

Lemma 2.1. The function $P_n(a)$ defined by

$$P_n(a) = a^{n+1}e^{-a}\left(\left(\frac{d}{da}\right)^n \frac{e^a-1}{a} - \frac{(-1)^{n+1}n!}{a^{n+1}}\right) \tag{2.5}$$

is a polynomial of degree n.

PROOF. Let $D = \frac{d}{da}$. Then $D^{n+1} = D(D^n)$ produces the recurrence

$$P_{n+1}(a) = aP_n'(a) + (a-n-1)P_n(a). \tag{2.6}$$

The initial condition $P_0(a) = 1$ and (2.6) show that P_n is a polynomial of degree n. □

Theorem 2.2. The polynomial

$$Q_n(a) := (-1)^n P_n(-a) \tag{2.7}$$

has positive integer coefficients, written as

$$Q_n(a) = \sum_{k=0}^{n} b_{n,k} a^k. \tag{2.8}$$

These coefficients satisfy

$$\begin{aligned} b_{n+1,0} &= (n+1)b_{n,0} \\ b_{n+1,k} &= (n+1-k)b_{n,k} + b_{n,k-1}, \quad 1 \leqslant k \leqslant n \\ b_{n+1,n+1} &= b_{n,n}. \end{aligned} \tag{2.9}$$

Moreover, the polynomial $Q_n(a)$ is given by

$$Q_n(a) = n! \sum_{k=0}^{n} \frac{a^k}{k!} \tag{2.10}$$

PROOF. The recurrence (2.6) yields

$$Q_{n+1}(a) = -aQ_n'(a) + (a+n+1)Q_n(a). \tag{2.11}$$

The recursion for the coefficients $b_{n,k}$ follows directly from here. Morover, it is clear that $b_{n,n} = 1$ and $b_{n,0} = n!$. A little experimentation suggets that $b_{n,k} = n!/k!$, and this can be established from (2.9). □

This proposition amounts to the evaluation of **3.351.1** in [**4**]:

$$\int_0^u x^n e^{ax}\, dx = \frac{(-1)^{n+1}\, n!}{a^{n+1}} + \frac{e^{au}}{a^{n+1}} \sum_{k=0}^{n} \frac{n!}{k!}(-1)^{n-k} u^k a^k. \tag{2.12}$$

The reader will find a proof of this formula in [**2**].

3. A new family of integrals

In this section we consider the family of integrals

$$q_n := \int_0^1 x^n e^{-x} \ln x\, dx, \tag{3.1}$$

and its companion

$$p_n := \int_0^1 x^n e^{-x}\, dx. \tag{3.2}$$

Lemma 3.1. The integrals p_n, q_n satisfy the recursion

$$p_{n+1} = (n+1)p_n - e^{-1} \tag{3.3}$$

$$q_{n+1} = (n+1)q_n + p_n \tag{3.4}$$

PROOF. Integrate by parts. □

The initial conditions are

$$p_0 = 1 - e^{-1} \text{ and } q_0 = \int_0^1 e^{-x} \ln x \, dx = \gamma - \text{Ei}(-1). \tag{3.5}$$

Here γ is Euler's constant defined by

$$\gamma := \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n \tag{3.6}$$

with integral representation

$$\gamma = \int_0^\infty e^{-x} \ln x \, dx \tag{3.7}$$

given as **4.331.1**. The reader will find in [**3**] a proof of this identity. The second term in (3.5) is converted into

$$\int_1^\infty e^{-x} \ln x \, dx = \int_1^\infty \frac{e^{-x}}{x} \, dx \tag{3.8}$$

and this last form is identified as Ei(−1), where Ei is the exponential integral defined by

$$\text{Ei}(z) = -\int_{-z}^\infty \frac{e^{-x}}{x} \, dx. \tag{3.9}$$

In the current context, the value of Ei(−1) will be simply one of the terms in the initial condition q_0.

We determine first an explicit expression for p_n. The recursion (3.3) shows the existence of integers a_n, b_n such that

$$p_n = a_n + b_n e^{-1}, \tag{3.10}$$

with $a_0 = 1$, $b_0 = -1$. From (3.3) we obtain

$$a_{n+1} + b_{n+1}e^{-1} = (n+1)a_n + [(n+1)b_n - 1]\, e^{-1}. \tag{3.11}$$

The irrationality of e produce the system

$$a_{n+1} = (n+1)a_n, \text{ with } a_0 = 1, \tag{3.12}$$

$$b_{n+1} = (n+1)b_n - 1, \text{ with } b_0 = -1. \tag{3.13}$$

The expression $a_n = n!$ follows directly from (3.12). To solve (3.13), define $B_n := b_n/n!$ and observe that

$$B_{n+1} = B_n - \frac{1}{(n+1)!}, \tag{3.14}$$

that telescopes to

$$b_n = -n! \sum_{k=0}^{n} \frac{1}{k!}. \tag{3.15}$$

We have shown:

Proposition 3.2. The integral p_n in (3.2) is given by

$$p_n = \int_0^1 x^n e^{-x}\, dx = \frac{n!}{e} \left(e - \sum_{k=0}^{n} \frac{1}{k!} \right). \tag{3.16}$$

We now determine a similar closed-form for q_n. The recursion (3.4) shows the existence of integers c_n, d_n, f_n such that

$$q_n = c_n + d_n e^{-1} + f_n q_0. \tag{3.17}$$

In order to produce a system similar to (3.12,3.13) we will assume that the constants 1, e^{-1} and $q_0 = -(\gamma + \mathrm{Ei}(-1))$ are linearly independent over $\mathbb{Q}$. Under this assumption (3.4) produces

$$c_{n+1} = (n+1)c_n + n!, \tag{3.18}$$

$$d_{n+1} = (n+1)c_n - n! \sum_{k=0}^{n} \frac{1}{k!}, \tag{3.19}$$

$$f_{n+1} = (n+1) f_n, \tag{3.20}$$

with the initial conditions $c_0 = 0$, $d_0 = 0$ and $f_0 = 1$.

The expression $f_n = n!$ follows directly from (3.20). To solve (3.18) and (3.19) we employ the following result established in [**1**].

Lemma 3.3. Let a_n, b_n and r_n be sequences with $a_n, b_n \neq 0$. Assume that z_n satisfies

$$a_n z_n = b_n z_{n-1} + r_n,\ n \geqslant 1 \tag{3.21}$$

with initial condition z_0. Then

$$z_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \left(z_0 + \sum_{k=1}^{n} \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} r_k \right). \tag{3.22}$$

We conclude that

$$c_n = n! \sum_{k=1}^{n} \frac{1}{k}, \tag{3.23}$$

and

$$d_n = -n! \sum_{k=1}^{n} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{j!}. \tag{3.24}$$

The expression for c_n shows that they coincide with the Stirling numbers of the first kind: $c_n = |s(n+1, 2)|$.

We have established

Proposition 3.4. The integral q_n in (3.1) is given by

$$q_n = \int_0^1 x^n e^{-x} \ln x \, dx = n! \left[\frac{1}{e} \sum_{k=1}^{n} \frac{1}{k} \left(e - \sum_{j=0}^{k-1} \frac{1}{j!} \right) + q_0 \right]. \tag{3.25}$$

EXAMPLE 3.1. The expressions for p_n and q_n provide the evaluation of **4.351.1** in [**4**]

$$\int_0^1 (1-x) e^{-x} \ln x \, dx = \frac{1-e}{e}, \tag{3.26}$$

by identifying the integral as $q_0 - q_1$. The recurrence (3.4) shows that

$$q_0 - q_1 = -p_0 = e^{-1} - 1, \tag{3.27}$$

as claimed.

EXAMPLE 3.2. The evaluation of **4.362.1** in [**4**]

$$\int_0^1 x e^x \ln(1-x) \, dx = \int_0^1 (1-t) e^{1-t} \ln t \, dt \tag{3.28}$$

is achieved by observing that this integral is $e(q_0 - q_1) = 1 - e$.

4. A parametric family

In this section we consider the evaluation of

$$P_n(a) := \int_0^1 x^n e^{-ax} \, dx \tag{4.1}$$

$$Q_n(a) := \int_0^1 x^n e^{-ax} \ln x \, dx. \tag{4.2}$$

The integrals q_n considered in Section 3 corresponds to the special case: $q_n = Q_n(1)$.
We now establish a recursion for Q_n by differentiating (4.2).

Lemma 4.1. The integral $Q_n(a)$ satisfies the relation

$$Q_{n+1}(a) = -\frac{d}{da} Q_n(a). \tag{4.3}$$

To obtain a closed-form expression for $Q_n(a)$ we need to determine the initial condition

$$Q_0(a) = \int_0^1 e^{-ax} \ln x \, dx. \tag{4.4}$$

This is expressed in terms of the *incomplete gamma function* defined in **8.350.1** by

$$\Gamma(a, x) := \int_x^\infty t^{a-1} e^{-t} \, dt. \tag{4.5}$$

Observe that $\Gamma(a, 0) = \Gamma(a)$, the usual gamma function.

Lemma 4.2. The initial condition $Q_0(a)$ is given by

$$Q_0(a) = \int_0^1 e^{-ax} \ln x \, dx = -\frac{1}{a} \left(\gamma + \ln a + \Gamma(0,a) \right). \tag{4.6}$$

Proof. The change of variables $t = ax$ yields

$$Q_0(a) = \frac{1}{a} \int_0^a e^{-t} \ln t \, dt - \frac{\ln a}{a} \left(1 - e^{-a} \right). \tag{4.7}$$

Then

$$\int_0^a e^{-t} \ln t \, dt = \int_0^\infty e^{-t} \ln t \, dt - \int_a^\infty e^{-t} \ln t \, dt. \tag{4.8}$$

The first integral is

$$\int_0^\infty e^{-t} \ln t \, dt = -\gamma, \tag{4.9}$$

that simply reflects the fact that $\gamma = -\Gamma'(1)$. Integrating by parts yields

$$\int_a^\infty e^{-t} \ln t \, dt = e^{-a} \ln a + \Gamma(0,a). \tag{4.10}$$

The formula for $Q_0(a)$ is established. □

We now determine a closed-form expression for $P_n(a)$ and $Q_n(a)$ following the procedure employed in Section 3.

Lemma 4.3. The integrals P_n and $Q_n(a)$ satisfy the recursion

$$P_{n+1}(a) = \frac{1}{a} \left((n+1) P_n(a) - e^{-a} \right) \tag{4.11}$$

$$Q_{n+1}(a) = \frac{1}{a} \left((n+1) Q_n(a) + P_n(a) \right). \tag{4.12}$$

The initial conditions are given by

$$P_0(a) = \frac{1}{a}(1 - e^{-a}), \text{ and } Q_0(a) = -\frac{1}{a}(\gamma + \Gamma(0,a) + \ln a). \tag{4.13}$$

Proof. Integrate by parts. □

We conclude that we can write

$$P_n(a) = A_n(a) - B_n(a) e^{-a}, \tag{4.14}$$

and

$$Q_n(a) = C_n(a) - D_n(a) e^{-a} - E_n(a)(\gamma + \Gamma(0,a) + \ln a). \tag{4.15}$$

Lemma 4.4. The recursions (4.11) and (4.12) imply that

$$\begin{aligned} A_{n+1}(a) &= \frac{1}{a}(n+1)A_n(a), \\ B_{n+1}(a) &= \frac{1}{a}\left[(n+1)B_n(a)+1\right], \\ C_{n+1}(a) &= \frac{1}{a}\left[(n+1)C_n(a)+A_n(a)\right], \\ D_{n+1}(a) &= \frac{1}{a}\left[(n+1)D_n(a)+B_n(a)\right], \\ E_{n+1}(a) &= \frac{1}{a}(n+1)E_n(a) \end{aligned} \tag{4.16}$$

with initial conditions

$$A_0(a) = B_0(a) = E_0(a) = \frac{1}{a} \text{ and } C_0(a) = D_0(a) = 0. \tag{4.17}$$

These recursion can now be solved as in Section 3 to produce a closed-form expression for the integrals $P_n(a)$ and $Q_n(a)$. We employ the notation

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{4.18}$$

for the harmonic numbers and

$$\text{Exp}_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \tag{4.19}$$

for the partial sums of the exponential function.

Theorem 4.5. Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$P_n(a) := \int_0^1 x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}} \left[1 - e^{-a}\text{Exp}_n(a)\right], \tag{4.20}$$

and

$$Q_n(a) := \int_0^1 x^n e^{-ax} \ln x\, dx = \frac{n!}{a^{n+1}} \left[H_n - G(a) - e^{-a} \sum_{k=1}^{n} \frac{1}{k}\text{Exp}_{k-1}(a)\right],$$

where $G(a) = -aQ_0(a) = \gamma + \Gamma(0, a) + \ln a$.

These expressions provide the evaluations of two integrals in [**4**].

Example 4.1. Formula **4.351.2** states that

$$\int_0^1 e^{-ax}(-ax^2 + 2x) \ln x\, dx = \frac{1}{a^2}\left[-1 + (1+a)e^{-a}\right]. \tag{4.21}$$

In order to verify this, observe that the stated integral is

$$-a\int_0^1 x^2 e^{-ax} \ln x\, dx + 2\int_0^1 x e^{-ax} \ln x\, dx = -aQ_2(a) + 2Q_1(a). \tag{4.22}$$

The expressions in Theorem 4.5 now complete the evaluation.

EXAMPLE 4.2. Formula **4.353.3** in [**4**] gives the value of

$$I_n(a) := \int_0^1 (-ax+n+1)x^n e^{-ax} \ln x \, dx. \tag{4.23}$$

Observe that

$$I_n(a) = -aQ_{n+1}(a) + (n+1)Q_n(a), \tag{4.24}$$

and using the recursion (4.12) we conclude that $I_n(a) = -P_n(a)$. The expression in Theorem 4.5 is precisely what appears in [**4**].

We conclude with the evaluation of a series shown to us by Tewodros Amdeberhan. Expand the exponential term in (4.21) and integrate term by term to obtain

$$\sum_{k=0}^{\infty} \frac{(-a)^k}{k!\,(n+1+k)^2} = \frac{n!}{a^{n+1}} \left(-\psi(n+1) + \ln a + \Gamma(0,a) + e^{-a} \sum_{k=0}^{n} \frac{1}{k} \text{Exp}_{k-1}(a) \right). \tag{4.25}$$

Here

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{4.26}$$

is the *digamma* function defined in **8.360.1** of [**4**]. the identity

$$\psi(n+1) = H_n - \gamma, \tag{4.27}$$

that is a direct consequence of the functional equation $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma'(1) = -\gamma$, was used to transform (4.25).

The identity (4.25) can be used to provide multiple expressions for the incomplete gamma function, such as

$$\int_a^{\infty} \frac{e^{-x}}{x} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k a^{n+1+k}}{n!\,k!\,(n+1+k)^2} + \psi(n+1) - \ln a - e^{-a} \sum_{k=1}^{n} \frac{\text{Exp}_{k-1}(a)}{k}, \tag{4.28}$$

and the special case for $n = 0$:

$$\int_a^{\infty} \frac{e^{-x}}{x} \, dx = -\gamma - \ln a + \sum_{k=0}^{\infty} \frac{(-1)^k a^{k+1}}{(k+1)!\,(k+1)}. \tag{4.29}$$

These issues will be explored in a future publication.

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