PROOF OF FORMULA 3.197.10

$$\int_0^1 \frac{x^{q-1}\,dx}{(1-x)^q(1+px)} = \frac{\pi}{(1+p)^q \sin \pi q}$$

Formula 3.197.4 gives

$$\int_0^1 x^{q-1}(1-x)^{-q}(1+px)^{-1}\,dx = B(q,1-q)\,{}_2F_1[1,q;1;-p].$$

The result now follows from $B(q,1-q) = \pi/\sin \pi q$ and ${}_2F_1[1,q;1;-p] = (1+p)^{-q}$. This last formula is given in 9.121.1.