PROOF OF FORMULA 3.197.11

$$\int_0^1 \frac{x^{p-1/2}\,dx}{(1-x)^p\,(1+qx)^p} = \frac{2\Gamma(p+\frac{1}{2})\Gamma(1-p)\ \cos^{2p}(z)\sin[(2p-1)z]}{\sqrt{\pi}\,(2p-1)\sin z}$$

where $z = \tan^{-1}(\sqrt{q})$.

The integral representation for the hypergeometric function is

$$ {}_2F_1\left[\alpha,\beta;\gamma;z\right] = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt.$$

Therefore

$$\int_0^1 \frac{x^{p-1/2}\,dx}{(1-x)^p\,(1+qx)^p} = B(p+\tfrac{1}{2},1-p){}_2F_1\left[p,p+\tfrac{1}{2};\tfrac{3}{2};-q\right].$$

Introduce the notation $n = 2p-1$ and define z by $q = \tan^2 z$. This gives

$$\int_0^1 \frac{x^{p-1/2}\,dx}{(1-x)^p\,(1+qx)^p} = B(p+\tfrac{1}{2},1-p){}_2F_1\left[\tfrac{n+1}{2},\tfrac{n+2}{2};\tfrac{3}{2};-\tan^2 z\right].$$

Formula 9.121.19 states that

$$ {}_2F_1\left[\tfrac{n+1}{2},\tfrac{n+2}{2};\tfrac{3}{2};-\tan^2 z\right] = \frac{\sin[(2p-1)z]\,\cos^{n+1} z}{(2p-1)\,\sin z}.$$

This is the result.