PROOF OF FORMULA 3.231.1

$$\int_0^1 \frac{x^{p-1} - x^{-p}}{1-x}\, dx = \pi \,\cot \pi p$$

Define

$$I(\epsilon) := \int_0^1 \frac{x^{p-1} - x^{-p}}{(1-x)^{1-\epsilon}}.$$

Then

$$\begin{aligned}
I(\epsilon) &= \int_0^1 x^{p-1}(1-x)^{\epsilon-1}\, dx - \int_0^1 x^{-p}(1-x)^{\epsilon-1}\, dx \\
&= B(p,\epsilon) - B(1-p,\epsilon) \\
&= \Gamma(\epsilon)\left(\frac{\Gamma(p)}{\Gamma(p+\epsilon)} - \frac{\Gamma(1-p)}{\Gamma(1-p+\epsilon)} \right) \\
&= -\frac{\Gamma(1+\epsilon)\Gamma(1-p+\epsilon)}{\Gamma(p+\epsilon)\Gamma(1-p)} \times \frac{\Gamma(p+\epsilon)-\Gamma(p)}{\epsilon} \\
&\quad -\frac{\Gamma(1+\epsilon)}{\Gamma(1-p+\epsilon)} \times \frac{\Gamma(1-p)-\Gamma(1-p+\epsilon)}{\epsilon}.
\end{aligned}$$

Now let $\epsilon \to 0$ to obtain

$$\int_0^1 \frac{x^{p-1} - x^{-p}}{1-x}\, dx = -\frac{\Gamma'(p)}{\Gamma(p)} + \frac{\Gamma'(1-p)}{\Gamma(1-p)} = -\psi(p) + \psi(1-p),$$

where ψ is the logarithmic derivative of the gamma function. The relation

$$\psi(x) - \psi(1-x) = -\pi \cot(\pi x)$$

that comes from logarithmic differentiation of

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

gives the result.