PROOF OF FORMULA 3.231.3

$$\int_0^1 \frac{x^p + x^{-p}}{x-1}\,dx = \frac{1}{p} - \pi\cot\pi p$$

Define

$$I(\epsilon) = -\int_0^1 x^p(1-x)^{-1+\epsilon}\,dx + \int_0^1 x^{-p}(1-x)^{-1+\epsilon}\,dx.$$

Then

$$\int_0^1 \frac{x^p + x^{-p}}{x-1}\,dx = \lim_{\epsilon\to 0} I(\epsilon).$$

Now,

$$\begin{aligned} I(\epsilon) &= B(1-p,\epsilon) - B(1+p,\epsilon) \\ &= \frac{\Gamma(1-p)\Gamma(\epsilon)}{\Gamma(1-p+\epsilon)} - \frac{\Gamma(1+p)\Gamma(\epsilon)}{\Gamma(1+p+\epsilon)} \\ &= \frac{\Gamma(1+\epsilon)}{\Gamma(1-p+\epsilon)\Gamma(1+p+\epsilon)} \times \\ & \quad \left(\Gamma(1-p)\frac{\Gamma(1+p+\epsilon)-\Gamma(1+p)}{\epsilon} - \Gamma(1+p)\frac{\Gamma(1-p+\epsilon)-\Gamma(1-p)}{\epsilon}\right). \end{aligned}$$

As $\epsilon \to 0$, the integral $I(\epsilon)$ converges to $\psi(1+p) - \psi(1-p)$, where $\psi = \Gamma'/\Gamma$ is the logarithmic derivative of the gamma function.

The result simplifies using the identities

$$\psi(1+p) = \frac{1}{p} + \psi(p) \text{ and } \psi(p) - \psi(1-p) = \pi\cot(\pi p).$$