PROOF OF FORMULA 3.231.4

$$\int_0^1 \frac{x^p - x^{-p}}{1+x}\,dx = \frac{1}{p} - \frac{\pi}{\sin \pi p}$$

The beta function

$$\beta(c) = \int_0^1 \frac{x^{c-1}\,dx}{1+x},$$

is related to the logarithmic derivative of the gamma function by

$$\beta(c) = \frac{1}{2}\left[\psi\left(\frac{c+1}{2}\right) - \psi\left(\frac{c}{2}\right)\right].$$

Therefore

$$\begin{aligned}
\int_0^1 \frac{x^p - x^{-p}}{1+x}\,dx &= \beta(1+p) - \beta(1-p) \\
&= \frac{1}{2}\left[\psi\left(1+\tfrac{p}{2}\right) - \psi\left(\tfrac{1}{2}+\tfrac{p}{2}\right) - \psi\left(1-\tfrac{p}{2}\right) + \psi\left(\tfrac{1}{2}-\tfrac{p}{2}\right)\right].
\end{aligned}$$

The result is simplified using the relations

$$\begin{aligned}
\psi(x+1) &= 1/x + \psi(x) \\
\psi(1-x) &= \psi(x) + \pi \cot \pi x \\
\psi(\tfrac{1}{2}+x) &= \psi(\tfrac{1}{2}-x) + \pi \tan \pi x/2.
\end{aligned}$$