PROOF OF FORMULA 3.234.1

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = \frac{\pi}{a^q} \cot \pi q$$

The change of variables $t = ax$ in the first integral and $x = at$ in the second one show that

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} + \frac{x^{-q}}{a-x} \right) dx = a^{-q} \left(\int_0^a \frac{t^{q-1}\, dt}{1-t} - \int_0^{1/a} \frac{t^{-q}\, dt}{1-t} \right).$$

Differentiation with respect to the parameter a shows that the sum of the two integrals is independent of a. Therefore

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} + \frac{x^{-q}}{a-x} \right) dx = a^{-q} \int_0^1 \frac{t^{q-1} - t^{-q}}{1-t}\, dt.$$

This integral is $\pi \cot \pi q$ as shown in formula 3.231.1.