PROOF OF FORMULA 3.247.1

$$\int_0^1 \frac{x^{a-1}(1-x)^{n-1}}{1-\xi x^b}\,dx = (n-1)!\sum_{k=0}^{\infty} \frac{\xi^k}{(a+bk)(a+bk+1)\cdots(a+bk+n-1)}$$

Expand the integrand to write

$$\int_0^1 \frac{x^{a-1}(1-x)^{n-1}}{1-\xi x^b}\,dx = \sum_{k=0}^{\infty} \xi^k \int_0^1 x^{a+bk-1}(1-x)^{n-1}\,dx.$$

The integral is a beta value and it yields

$$\int_0^1 \frac{x^{a-1}(1-x)^{n-1}}{1-\xi x^b}\,dx = \sum_{k=0}^{\infty} \xi^k \frac{\Gamma(a+bk)\Gamma(n)}{\Gamma(a+bk+n)}.$$

The result now follows from

$$\Gamma(a+n) = \Gamma(a)(a)_n.$$