PROOF OF FORMULA 3.248.2

$$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}\,n!}{2\Gamma(n+3/2)} = \frac{(2n)!!}{(2n+1)!!}$$

Let $t = x^2$ to obtain

$$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\int_0^1 t^n (1-t)^{-1/2}\,dt.$$

Now use the representation

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

with $x = n+1$ and $y = 1/2$ to obtain

$$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{1}{2}B(n+1,1/2) = \frac{\Gamma(n+1)\Gamma(1/2)}{2\Gamma(n+3/2)},$$

where we have used

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The value $\Gamma(1/2) = \sqrt{\pi}$ and the functional equation

$$\Gamma(x+1) = x\Gamma(x)$$

give the identity

$$\Gamma(n+\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2^{n+1}}(2n+1)!!.$$

The relation

$$(2n)!! = 2^n n!$$

is useful in the simplifications.