PROOF OF FORMULA 3.248.3

$$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1!!}{(2n)!!}\frac{\pi}{2}$$

The change of variables $t = x^2$ gives

$$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\int_0^1 t^{n-1/2}(1-t)^{-1/2}dt$$

and using the formula

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

with $x = n+\frac{1}{2}$ and $y = \frac{1}{2}$, we get

$$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1}{2}B\left(n+\tfrac{1}{2},\tfrac{1}{2}\right).$$

Now use

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and the expression

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)!!}{2^n}$$

to obtain the result.