

PROOF OF FORMULA 3.248.6

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{b+ax^2}} = \begin{cases} \frac{2}{\sqrt{b-a}} \tan^{-1}(\sqrt{b/a}-1) & \text{if } a < b \\ \frac{2}{\sqrt{a}} & \text{if } a = b \\ \frac{1}{\sqrt{a-b}} \ln\left(\frac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right) & \text{if } a > b \end{cases}$$

Define

$$q(a, b) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{b+ax^2}}.$$

The change of variables $x = \sqrt{bt}/\sqrt{a}$ gives

$$q(a, b) = 2\sqrt{a} \int_0^{\infty} \frac{dt}{(a+bt^2)\sqrt{1+t^2}}.$$

Now let $t = \tan \theta$ to write the integral in trigonometric form

$$q(a, b) = 2\sqrt{a} \int_0^{\pi/2} \frac{\cos \theta d\theta}{a \cos^2 \theta + b \sin^2 \theta}.$$

The final change of variable $u = \sin \theta$ produces

$$q(a, b) = 2\sqrt{a} \int_0^1 \frac{du}{a + (b-a)u^2}.$$

This is elementary.

The special case $a = 3$ and $b = 4$ gives 3.248.4.