PROOF OF FORMULA 3.265

$$\begin{aligned}\int_0^1 \frac{1-x^{a-1}}{1-x}\,dx &= \psi(a)+\gamma \\ &= \psi(1-a)+\gamma-\pi\cot(\pi a)\end{aligned}$$

This is special case of the formula

$$\int_0^1 \frac{x^{p-1}-x^{q-1}}{1-x}\,dx = \psi(q)-\psi(p).$$

In order to prove this consider first the integral

$$I(\epsilon) = \int_0^1 x^{p-1}(1-x)^{\epsilon-1}\,dx - \int_0^1 x^{q-1}(1-x)^{\epsilon-1}\,dx$$

that avoids the apparent singularity at $x=1$. The integral $I(\epsilon)$ can be expressed as

$$\begin{aligned}I(\epsilon) &= B(p,\epsilon)-B(q,\epsilon) \\ &= \Gamma(\epsilon)\left(\frac{\Gamma(p)}{\Gamma(p+\epsilon)}-\frac{\Gamma(q)}{\Gamma(q+\epsilon)}\right) \\ &= \Gamma(1+\epsilon)\left(\frac{\Gamma(p)-\Gamma(p+\epsilon)}{\epsilon}\frac{1}{\Gamma(p+\epsilon)}-\frac{\Gamma(q)-\Gamma(q+\epsilon)}{\epsilon}\frac{1}{\Gamma(q+\epsilon)}\right).\end{aligned}$$

The result is now obtained by letting $\epsilon\to 0$.