PROOF OF FORMULA 3.312.3

$$\int_0^\infty (1-e^{-x})^{\nu-1}(1-be^{-x})^{-\rho}e^{-\mu x}\,dx = B(\mu,\nu)\,{}_2F_1\left[\rho,\mu;\mu+\nu;b\right]$$

The change of variables $t = e^{-x}$ gives

$$\int_0^\infty (1-e^{-x})^{\nu-1}(1-be^{-x})^{-\rho}e^{-\mu x}\,dx = \int_0^1 t^{\mu-1}(1-t)^{\nu-1}(1-bt)^{-\rho}\,dt.$$

The result now follows from the integral representation of the hypergeometric function

$${}_2F_1[\alpha,\beta;\gamma;z] = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt.$$