PROOF OF FORMULA 3.331.3

$$\int_0^\infty (1-e^{-x})^{\nu-1} e^{be^{-x}-\mu x}\,dx = B(\mu,\nu)e^{b/2}b^{-(\mu+\nu)/2}\,M_{\frac{\nu-\mu}{2},\frac{\nu+\mu-1}{2}}(b)$$

The *Whittaker function* is defined by the integral representation

$$M_{a,b}(z) = \frac{z^{b+1/2}}{2^{2b}B(a+b+\frac{1}{2},b-a+\frac{1}{2})}\int_{-1}^1 (1+t)^{b-a-1/2}(1-t)^{b+a-1/2}\,e^{zt/2}\,dt$$

The change of variables $t=e^{-x}$ gives

$$\int_0^\infty (1-e^{-x})^{\nu-1}e^{be^{-x}-\mu x}\,dx = \int_0^1 (1-t)^{\nu-1}e^{bt}t^{\mu-1}\,dt.$$

The further change of variables $s=2t+1$ produces

$$\int_0^\infty (1-e^{-x})^{\nu-1}e^{be^{-x}-\mu x}\,dx = e^{b/2}2^{-(\mu+\nu-1)}\int_{-1}^1 (1+s)^{\mu-1}(1-s)^{\nu-1}\,e^{bs/2}\,ds.$$

This is the formula, with $a=(\mu+\nu-1)/2$ and $a=(\nu-\mu)/2$ in the definition of the Whittaker function.