PROOF OF FORMULA 3.339

$$\int_0^\pi e^{z\ \cos x}\,dx = \pi I_0(z)$$

Expand the exponential to obtain

$$\int_0^\pi e^{z\ \cos x}\,dx = \sum_{j=0}^{\infty} \frac{z^j}{j!} \int_0^\pi \cos^j x\,dx.$$

The integral vanishes for j odd by mmetry around $x = \pi/2$. In the case of even exponent $j = 2k$, Wallis's formula gives

$$\int_0^\pi \cos^{2k} x\,dx = \frac{(2k)!}{k!^2}\frac{\pi}{2^{2k}}.$$

Therefore

$$\int_0^\pi e^{z\ \cos x}\,dx = \pi \sum_{k=0}^{\infty} \frac{1}{k!^2}\left(\frac{z}{2}\right)^{2k}.$$

This is the power seties expansion of the Bessel function I_0.