

FORMULA 3.381.2

$$\begin{aligned} \int_0^u x^{p-1} e^{-x} dx &= \sum_{k=0}^{\infty} (-1)^k \frac{u^{p+k}}{k!(p+k)} \\ &= e^{-u} \sum_{k=0}^{\infty} \frac{u^{p+k}}{p(p+1)\cdots(p+k)} \end{aligned}$$

Expanding the integrand yields

$$\begin{aligned} \int_0^u x^{p-1} e^{-x} dx &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^u x^{p+k-1} dx \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{u^{p+k}}{k!(p+k)}. \end{aligned}$$

To check the second identity, let $t = u - x$ to obtain

$$\begin{aligned} \int_0^u x^{p-1} e^{-x} dx &= e^{-u} \int_0^u (u-t)^{p-1} e^{-t} dt \\ &= e^{-u} \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^u t^k (u-t)^{p-1} dt. \end{aligned}$$

The inner integral is simplified via $t = us$ to get

$$\int_0^u t^k (u-t)^{p-1} dt = u^{p+k} \int_0^1 s^k (1-s)^{p-1} ds.$$

The integral is identified as $B(k+1, p)$ and the result follows from

$$B(k+1, p) = \frac{\Gamma(k+1)\Gamma(p)}{\Gamma(k+1+p)} = \frac{k!(p-1)!}{(k+p)!}.$$