# PROOF OF FORMULA 3.411.1

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{ax}-1} = \frac{\Gamma(s)\,\zeta(s)}{a^s}$$

Let $t = ax$ to get

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{ax}-1} = a^{-s}\int_0^\infty \frac{t^{s-1}\,dt}{e^t-1}.$$

Observe that

$$\frac{1}{e^t-1} = \frac{e^{-t}}{1-e^{-t}} = \sum_{k=0}^\infty e^{-(k+1)t},$$

and integrating term by term we have

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{ax}-1} = a^{-s}\sum_{k=0}^\infty \int_0^\infty t^{s-1}e^{-(1+k)t}\,dt.$$

The change of variables $u = t(1+k)$ gives the result.