PROOF OF FORMULA 3.411.10

$$\int_0^\infty \frac{xe^{-2x}\,dx}{e^{-x}+1} = 1 - \frac{\pi^2}{12}$$

The integral is

$$\int_0^\infty \frac{xe^{-2x}\,dx}{e^{-x}+1} = \int_0^\infty \frac{xe^{-x}\,dx}{1+e^x}.$$

Formula 3.411.8 states that

$$\int_0^\infty \frac{x^{n-1}e^{-px}\,dx}{1+e^x} = (n-1)!\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k+p)^n}.$$

The special case considered here is $n=2$ and $p=1$, so that

$$\int_0^\infty \frac{xe^{-2x}\,dx}{e^{-x}+1} = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k+1)^2}.$$

The standard even-odd index trick gives

$$\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k+1)^2} = \sum_{k=1}^\infty \frac{(-1)^k}{k^2} + 1 = 1 - \frac{\pi^2}{12}.$$