PROOF OF FORMULA 3.411.11

$$\int_0^\infty \frac{xe^{-3x}\,dx}{e^{-x}+1} = \frac{\pi^2}{12} - \frac{3}{4}$$

Expand the denominator of the integrand in a geometric series to obtain

$$\int_0^\infty \frac{xe^{-3x}\,dx}{e^{-x}+1} = \sum_{k=0}^\infty (-1)^k \int_0^\infty xe^{-(k+3)x}\,dx.$$

The change of variables $t = (k+3)x$ gives

$$\int_0^\infty \frac{xe^{-3x}\,dx}{e^{-x}+1} = \sum_{k=0}^\infty \frac{(-1)^k}{(k+3)^2} \int_0^\infty te^{-t}\,dt.$$

The integral is $\Gamma(2) = 1$, so that

$$\int_0^\infty \frac{xe^{-3x}\,dx}{e^{-x}+1} = -\sum_{k=3}^\infty \frac{(-1)^k}{k^2}.$$

Now use

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12},$$

to obtain the result.