PROOF OF FORMULA 3.411.7

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}\,dx}{1-e^{-bx}} = \frac{\Gamma(\nu)}{b^\nu}\zeta\left(\nu, \frac{\mu}{b}\right)$$

Expand the integrand to obtain

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}\,dx}{1-e^{-bx}} = \sum_{k=0}^\infty \int_0^\infty x^{\nu-1}e^{-(\mu+kb)x}\,dx.$$

The change of variables $t = (\mu + kb)x$ yields

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}\,dx}{1-e^{-bx}} = \sum_{k=0}^\infty \frac{1}{(\mu+kb)^\nu} \int_0^\infty t^{\nu-1}e^{-t}\,dt.$$

The integral is $\Gamma(\nu)$ and the series is written in terms of the Hurwitz zeta function

$$\zeta(s,q) = \sum_{k=0}^\infty \frac{1}{(k+q)^s},$$

as indicated in the formula.