PROOF OF FORMULA 3.411.9

$$\int_0^\infty \frac{xe^{-x}\,dx}{e^x-1} = \frac{\pi^2}{6} - 1$$

The integral is

$$\int_0^\infty \frac{xe^{-x}\,dx}{e^x-1} = \int_0^\infty \frac{xe^{-2x}\,dx}{1-e^{-x}}.$$

Formula 3.411.7 states that

$$\int_0^\infty \frac{x^{\nu-1}\,e^{-\mu x}\,dx}{1-e^{-bx}} = \frac{\Gamma(\nu)}{b^\nu}\zeta\left(\nu, \frac{\mu}{b}\right).$$

The case considered here is $\nu = 2$, $b = 1$ and $\mu = 2$. Thus,

$$\int_0^\infty \frac{xe^{-x}\,dx}{e^x-1} = \Gamma(2)\zeta(2,2).$$

The Hurwitz zeta function gives

$$\zeta(2,2) = \sum_{n=0}^\infty \frac{1}{(n+2)^2} = \sum_{n=1}^\infty \frac{1}{n^2} - 1.$$

The value $\zeta(2) = \pi^2/6$ completes the evaluation.