PROOF OF FORMULA 3.418.1

$$\int_0^\infty \frac{x\,dx}{e^x + e^{-x} - 1} = \frac{1}{3}\left[\psi'\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right]$$

The change of variable $t = e^{-x}$ gives

$$\int_0^\infty \frac{x\,dx}{e^x + e^{-x} - 1} = -\int_0^1 \frac{\ln t\,dt}{t^2 - t + 1}.$$

The integral representation of the *dilogarithm function* gives

$$\int_0^1 \frac{\ln t\,dt}{t+b} = \text{Li}_2(-1/b).$$

Factoring the quadratic and denoting $r_{1,2} = \frac{1}{2}(1 \pm i\sqrt{3})$, it follows that

$$\int_0^1 \frac{\ln t\,dt}{t^2 - t + 1} = \frac{1}{r_2 - r_1}\left(\text{Li}_2(-1/r_1) - \text{Li}_2(-1/r_2)\right).$$

Using the series representation of the dilogarithm gives

$$\int_0^1 \frac{\ln t\,dt}{t^2 - t + 1} = -\frac{2}{\sqrt{3}} \sum_{k=1}^\infty \frac{\sin(\pi k/3)}{k^2}.$$

The periodicity of the values of $\sin(\pi k/3)$ and the formula

$$\psi'(x) = -\sum_{k=0}^\infty \frac{1}{(x+k)^2}$$

show that the series is given by

$$\psi'\left(\tfrac{1}{6}\right) + \psi'\left(\tfrac{1}{3}\right) - \psi'\left(\tfrac{2}{3}\right) - \psi'\left(\tfrac{5}{6}\right).$$

The identities

$$\psi(1-x) = \psi(x) + \pi\cot\pi x \text{ and } \psi(2x) = \tfrac{1}{2}(\psi(x) + \psi(x + \tfrac{1}{2})) + \log 2$$

are use to simplify the result.