PROOF OF FORMULA 3.435.4

$$\int_0^\infty \left(e^{-\mu x} - \frac{1}{1+ax} \right) \frac{dx}{x} = \ln \frac{a}{\mu} - \gamma$$

The change of variables $t = \mu x$ gives

$$\int_0^\infty \left(e^{-\mu x} - \frac{1}{1+ax} \right) \frac{dx}{x} = \int_0^\infty \left(e^{-t} - \frac{1}{1+bt} \right) \frac{dt}{t},$$

with $b = a/\mu$. This can be written as

$$\int_0^\infty \left(e^{-t} - \frac{1}{1+t} \right) \frac{dt}{t} + \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{1+bt} \right) \frac{dt}{t}.$$

Formula 3.435.3 states that the first integral is $-\gamma$. The partial fraction decomposition

$$\left(\frac{1}{1+t} - \frac{1}{1+bt} \right) \frac{dt}{t} = \frac{b}{1+bt} - \frac{1}{1+t}$$

shows that the second integral is

$$\lim_{t \to \infty} \ln \frac{1+bt}{1+t} = \ln b.$$

Now replace to obtain the result.