

PROOF OF FORMULA 3.451.2

$$\int_0^{\infty} x e^{-x} \sqrt{1 - e^{-2x}} dx = \frac{\pi}{8} (1 + 2 \ln 2)$$

The change of variables $t = 2x$ shows that

$$\int_0^{\infty} x e^{-x} \sqrt{1 - e^{-2x}} dx = -\frac{1}{4} h'(\frac{1}{2}),$$

where

$$h(a) = \int_0^{\infty} e^{-ax} \sqrt{1 - e^{-x}} dx.$$

The change of variables $t = e^{-x}$ gives

$$h(a) = \int_0^1 t^{a-1} (1-t)^{1/2} dt = B(a, \frac{3}{2}).$$

Differentiation yields

$$h'(a) = h(a) [\psi(a) - \psi(a + \frac{3}{2})],$$

where $\psi = \Gamma'/\Gamma$ is the polygamma function. Therefore

$$\int_0^{\infty} x e^{-x} \sqrt{1 - e^{-2x}} dx = -\frac{1}{4} h(\frac{1}{2}) [\psi(\frac{1}{2}) - \psi(2)]$$

The value $\psi(\frac{1}{2}) = -(\gamma + 2 \ln 2)$ and $\psi(2) = -\gamma + 1$ give the result. To obtain the expression for $\psi(\frac{1}{2})$, differentiate the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}),$$

to produce

$$2\psi(2z) = 2 \ln 2 + \psi(z) + \psi(z + \frac{1}{2}).$$

The simplification also makes use of the relation

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}.$$