PROOF OF FORMULA 3.458.2

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{xe^x\,dx}{(a+e^x)^{\nu+1}} &= \frac{1}{\nu a^\nu}\left[\ln a - \gamma - \psi(\nu)\right] \\ &= \frac{1}{\nu a^\nu}\left[\ln a - \sum_{k=1}^{\nu-1}\frac{1}{k}\right] \qquad \text{if } \nu \in \mathbb{N}\end{aligned}$$

Define

$$f(b) = \int_0^\infty \frac{t^b\,dt}{(a+t)^{\nu+1}},$$

and observe that

$$f'(b) = \int_0^\infty \frac{t^b\,\ln t\,dt}{(a+t)^{\nu+1}}.$$

The change of variables $t = e^x$ gives

$$\int_{-\infty}^{\infty} \frac{xe^x\,dx}{(a+e^x)^{\nu+1}} = \int_0^\infty \frac{\ln t\,dt}{(a+t)^{\nu+1}},$$

so that the requested integral is $f'(0)$.

Observe next that

$$f(b) = a^{b-\nu}\int_0^\infty \frac{s^b\,ds}{(1+s)^{\nu+1}} = a^{b-\nu}B(b+1,\nu-b) = a^{b-\nu}\frac{\Gamma(b+1)\Gamma(\nu-b)}{\Gamma(\nu+1)}.$$

Logarithmic differentiation gives

$$f'(b) = [\ln a + \psi(b+1) - \psi(\nu-b)]f(b).$$

Therefore

$$f'(0) = \frac{\ln a - \gamma - \psi(\nu)}{\nu a^\nu},$$

as claimed.