

PROOF OF FORMULA 3.466.2

$$\int_0^\infty \frac{x^2 e^{-\mu^2 x^2} dx}{x^2 + b^2} = \frac{\sqrt{\pi}}{2\mu} - \frac{\pi b}{2} e^{b^2 \mu^2} [1 - \operatorname{erf}(b\mu)]$$

Start with

$$e^{-b^2 \mu^2} \int_0^\infty \frac{x^2 e^{-\mu^2 x^2} dx}{x^2 + b^2} = \int_0^\infty \frac{x^2 e^{-\mu^2(x^2+b^2)} dx}{x^2 + b^2}.$$

Now observe that

$$\frac{\partial}{\partial \mu} \left(\int_0^\infty \frac{x^2 e^{-\mu^2(x^2+b^2)} dx}{x^2 + b^2} \right) = -2\mu e^{-\mu^2 b^2} \int_0^\infty x^2 e^{-\mu^2 x^2} dx.$$

The change of variables $t = \mu x$ gives

$$\frac{\partial}{\partial \mu} \left(\int_0^\infty \frac{x^2 e^{-\mu^2(x^2+b^2)} dx}{x^2 + b^2} \right) = -\frac{2e^{-\mu^2 b^2}}{\mu^2} \int_0^\infty t^2 e^{-t^2} dt.$$

Integration by parts shows that the last integral is $\sqrt{\pi}/4$, therefore

$$\frac{\partial}{\partial \mu} \left(\int_0^\infty \frac{x^2 e^{-\mu^2(x^2+b^2)} dx}{x^2 + b^2} \right) = -\frac{\sqrt{\pi}}{2} \frac{e^{-\mu^2 b^2}}{\mu^2}.$$

Now integrate from μ to ∞ and use formula 3.461.5:

$$\int_\mu^\infty e^{-b^2 x^2} \frac{dx}{x^2} = \frac{1}{\mu} e^{-b^2 \mu^2} - \sqrt{\pi} b [1 - \operatorname{erf}(b\mu)]$$

to obtain the result.