PROOF OF FORMULA 3.476.2

$$\int_0^\infty \left[\exp(-x^p) - \exp(-x^q) \right] \frac{dx}{x} = \frac{p-q}{pq}\gamma$$

Let $u = x^p$ to obtain

$$\int_0^\infty \left[\exp(-x^p) - \exp(-x^q) \right] \frac{dx}{x} = \frac{1}{p} \int_0^\infty \left[e^{-u} - e^{-u^{p/q}} \right] \frac{du}{u}.$$

Use the representation

$$\gamma = - \int_0^\infty \left[e^{-u} - \frac{1}{1+u} \right] \frac{du}{u}$$

to obtain

$$\int_0^\infty \left[\exp(-x^p) - \exp(-x^q) \right] \frac{dx}{x} = -\frac{\gamma}{p} + \frac{1}{q} \int_0^\infty \left[\frac{1}{1+v^{p/q}} - e^{-v} \right].$$

It follows that

$$\int_0^\infty \left[\exp(-x^p) - \exp(-x^q) \right] \frac{dx}{x} = -\frac{\gamma}{p} + \frac{\gamma}{q} + \frac{1}{q} \int_0^\infty \frac{1 - v^{p/q-1}}{(1+v^{p/q})(1+v)} \, dv.$$

Split the last integral over $[0, 1]$ and $[1, \infty)$ and make the change of variables $v \mapsto 1/v$ in the second part to conclude that the whole integral vanishes.