PROOF OF FORMULA 3.511.2

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tan\left(\frac{\pi a}{2b}\right)$$

Write the integrand in exponential form to obtain

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \int_0^\infty \frac{e^{(a-b)x} - e^{-(a+b)x}}{1-e^{-2ax}}\,dx.$$

Expanding the integrand in series yields

$$\begin{aligned}\int_0^\infty \frac{e^{(a-b)x} - e^{-(a+b)x}}{1-e^{-2ax}}\,dx &= \sum_{k=0}^\infty \int_0^\infty \left(e^{-(b-a+2bk)x} - e^{-(b+a+2bk)x}\right)dx \\ &= \sum_{k=0}^\infty \left(\frac{1}{b-a+2bk} - \frac{1}{b+a+2bk}\right).\end{aligned}$$

The result now follows from the expansion

$$\tan\left(\frac{\pi x}{2}\right) = \frac{4x}{\pi}\sum_{k=1}^\infty \frac{1}{(2k-1)^2 - x^2}.$$