PROOF OF FORMULA 3.511.3

$$\int_0^\infty \frac{\sinh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\sec\left(\frac{\pi a}{2b}\right) - \frac{1}{b}\beta\left(\frac{a+b}{2b}\right)$$

Let $t = bx$ and write $c = a/b$. The formula is equivalent to

$$\int_0^\infty \frac{\sinh ct}{\cosh t}\,dt = \frac{\pi}{2}\sec\left(\frac{\pi c}{2}\right) - \beta\left(\frac{1+c}{2}\right).$$

Write the hyperbolic functions as exponential to obtain

$$\int_0^\infty \frac{\sinh ct}{\cosh t}\,dt = \int_0^\infty \frac{e^{(c-1)t} - e^{-(c+1)t}}{1+e^{-2t}}\,dt.$$

The change of variables $u = e^{-2t}$ gives

$$\int_0^\infty \frac{e^{(c-1)t} - e^{-(c+1)t}}{1+e^{-2t}}\,dt = \frac{1}{2}\int_0^1 \frac{u^{(1-c)/2-1} - u^{(1+c)/2-1}}{1+u}\,du.$$

From the representation

$$\beta(a) = \int_0^1 \frac{u^{a-1}}{1+u}\,du,$$

we conclude that

$$\int_0^\infty \frac{\sinh ct}{\cosh t}\,dt = \frac{1}{2}\left[\beta\left(\frac{1-c}{2}\right) = \beta\left(\frac{1+c}{2}\right)\right].$$

The result now follows from the identity

$$\beta(a) = \frac{1}{2}\left(\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{a}{2}\right)\right),$$

and

$$\psi(1-x) = \psi(x) + \pi\cot\pi x.$$