PROOF OF FORMULA 3.521.3

$$\int_1^\infty \frac{dx}{x\,\sinh ax} = -2\sum_{k=0}^\infty \mathrm{Ei}\left(-(2k+1)a\right)$$

Write the integral as

$$\int_1^\infty \frac{dx}{x\,\sinh ax} = 2\int_1^\infty \frac{e^{-ax}}{x}\,\frac{dx}{1-e^{-2ax}}.$$

Expand the integrand as a geometric series to produce

$$\int_1^\infty \frac{dx}{x\,\sinh ax} = 2\sum_{k=0}^\infty \int_1^\infty \frac{e^{-(2k+1)ax}}{x}\,dx$$

$$= 2\sum_{k=0}^\infty \int_{(2k+1)a}^\infty \frac{e^{-t}}{t}\,dt.$$

The result now follows from the definition of the exponential integral

$$\mathrm{Ei}(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}\,dt.$$