PROOF OF FORMULA 3.551.3

$$\int_0^\infty x^{\mu-1}e^{-bx}\coth x\,dx = \Gamma(\mu)\left[2^{1-\mu}\zeta\left(\mu,\tfrac{b}{2}\right)-b^{-\mu}\right]$$

Use

$$\coth x = 1+\frac{2e^{-x}}{e^x+e^{-x}}$$

to obtain

$$\int_0^\infty x^{\mu-1}e^{-bx}\coth x\,dx = \int_0^\infty x^{\mu-1}e^{-bx}\,dx + 2\int_0^\infty \frac{x^{\mu-1}e^{-(b+2)x}\,dx}{1-e^{-2x}}.$$

The change of variables $t=2x$ in the second integral and the representation

$$\zeta(z,q)=\frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}e^{-qt}\,dt}{1-e^{-t}}$$

produce

$$\int_0^\infty x^{\mu-1}e^{-bx}\coth x\,dx = b^{-\mu}\Gamma(\mu)+2^{1-\mu}\Gamma(\mu)\zeta(\mu,\tfrac{b}{2}+1).$$

The result now follows from the elementary identity

$$\zeta(\mu,q+1)=\zeta(\mu,q)-q^{-\mu}.$$