PROOF OF FORMULA 3.551.5

$$\int_0^1 \frac{e^{-bx}}{x} \sinh ax \, dx = \frac{1}{2} \left[\ln \frac{b+a}{b-a} + \text{Ei}(a-b) - \text{Ei}(-a-b) \right]$$

The exponential integral is defined by

$$\text{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

for $z < 0$ and by the corresponding principal value for $z \geq 0$. The requested integral is $\lim\limits_{\epsilon \to 0} I_\epsilon$, where

$$I_\epsilon = \int_\epsilon^1 \frac{e^{-bx}}{x} \sinh ax \, dx.$$

Now observe that

$$\begin{aligned} 2I_\epsilon &= \int_\epsilon^1 \frac{e^{-(b-a)x}}{x} \, dx - \int_\epsilon^1 \frac{e^{-(b+a)x}}{x} \, dx \\ &= \int_{\epsilon(b-a)}^{b-a} \frac{e^{-t}}{t} dt - \int_{\epsilon(b+a)}^{b+a} \frac{e^{-t}}{t} dt \\ &= \int_{\epsilon(b-a)}^{\infty} \frac{e^{-t}}{t} \, dt - \int_{b-a}^{\infty} \frac{e^{-t}}{t} \, dt - \int_{\epsilon(b+a)}^{\infty} \frac{e^{-t}}{t} \, dt + \int_{b+a}^{\infty} \frac{e^{-t}}{t} \, dt \\ &= \int_{\epsilon(b-a)}^{\epsilon(b+a)} \frac{e^{-t}}{t} dt + \int_{\epsilon(b-a)}^{\epsilon(b+a)} \frac{dt}{t} + \text{Ei}(a-b) - \text{Ei}(-a-b). \end{aligned}$$

As $\epsilon \to 0$, the first term vanishes and the second one converges to the logarithmic part appearing in the answer.