PROOF OF FORMULA 3.551.8

$$\int_0^\infty xe^{-x} \coth x \, dx = \frac{\pi^2}{4} - 1$$

The integral is

$$\int_0^\infty xe^{-x} \coth x \, dx = \int_0^\infty xe^{-x} \frac{1+e^{-2x}}{1-e^{-2x}} \, dx.$$

The change of variables $t = 2x$ gives

$$\int_0^\infty xe^{-x} \coth x \, dx = \frac{1}{4} \int_0^\infty \frac{te^{-t/2} + te^{-3t/2}}{1-e^{-t}} \, dt.$$

Employ the integral representation

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}e^{-qt}}{1-e^{-t}} \, dt$$

to conclude that

$$\int_0^\infty xe^{-x} \coth x \, dx = \frac{\Gamma(2)}{4}\zeta(1, \tfrac{1}{2}) + \frac{\Gamma(2)}{4}\zeta(1, \tfrac{3}{2}).$$

The Hurwitz zeta function is

$$\zeta(z,q) = \sum_{n=0}^\infty \frac{1}{(n+q)^z}$$

and this reduces the previous sums to

$$\int_0^\infty xe^{-x} \coth x \, dx = 2 \sum_{n=0}^\infty \frac{1}{(2n+1)^2} - 1.$$

The value

$$\sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

gives the result.