PROOF OF FORMULA 3.551.9

$$\int_0^\infty e^{-bx} \tanh x \, \frac{dx}{x} = \ln \frac{b}{4} + 2 \ln \left(\frac{\Gamma(b/4)}{\Gamma(b/4+1/2)} \right)$$

Write the integral as

$$\int_0^\infty e^{-bx} \tanh x \, \frac{dx}{x} = \int_0^\infty \frac{e^{-bx}(1-e^{-2x})}{1+e^{-2x}} \, \frac{dx}{x}.$$

The change of variables $t = 2x$ gives

$$\int_0^\infty e^{-bx} \tanh x \, \frac{dx}{x} = \int_0^\infty \frac{e^{-bt/2} - e^{-(b/2+1)t}}{1+e^{-t}} \, \frac{dt}{t}.$$

Entry 3.411.28 states that

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{e^{-x}+1} \, \frac{dx}{x} = \ln \left(\frac{\Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\nu+1}{2}\right)} \right).$$

Therefore

$$\int_0^\infty e^{-bx} \tanh x \, \frac{dx}{x} = \ln \left(\frac{\Gamma\left(\frac{b}{4}\right) \Gamma\left(\frac{b}{4}+1\right)}{\Gamma\left(\frac{b}{4}+\frac{1}{2}\right) \Gamma\left(\frac{b}{4}+\frac{1}{2}\right)} \right).$$

This reduces to the stated answer.