PROOF OF FORMULA 3.552.5

$$\int_0^\infty \frac{x^2 e^{-2nx}}{\sinh x}\, dx = 4 \sum_{k=n}^\infty \frac{1}{(2k+1)^3}$$

Write the integral as

$$\int_0^\infty \frac{x^2 e^{-2nx}}{\sinh x}\, dx = 2 \int_0^\infty \frac{x^2 e^{-(2n+1)x}}{1-e^{-2x}}\, dx$$

and expand the integrand as a geometric series to obtain

$$\int_0^\infty \frac{x^2 e^{-2nx}}{\sinh x}\, dx = 2 \sum_{k=0}^\infty \int_0^\infty x^2 e^{-(2n+2k+1)x}\, dx.$$

The result follows from the change of variables $u = (2n+2k+1)x$ and the value

$$\int_0^\infty u^2 e^{-u}\, du = \Gamma(3) = 2.$$