PROOF OF FORMULA 4.224.7

$$\int_0^{\pi/2} \ln^2 \sin x \, dx = \frac{\pi}{2} \left[\ln^2 2 + \frac{\pi^2}{12} \right]$$

The recurrence presented here is due to M. G. Beumer, Amer. Math. Monthly 68(1961), 645-647. Let

$$S_n := \int_0^{\pi/2} \ln^n \sin x \, dx$$

and define the Dirichlet series

$$X(s) = \sum_{j=0}^{\infty} \frac{2^{-2j} \binom{2j}{j}}{(2j+1)^s}.$$

Using

$$\int_0^{\infty} x^{s-1} e^{-(2n+1)x} \, dx = \frac{\Gamma(s)}{(2n+1)^s},$$

the binomial theorem

$$(1-t)^{-1/2} = \sum_{k=0}^{\infty} 2^{-2k} \binom{2k}{k} t^k$$

yields

$$X(s)\Gamma(s) = \int_0^{\infty} \frac{x^{s-1} \, dx}{\sqrt{e^{2x}-1}}.$$

The change of variables $e^{-x} = \sin\theta$ shows that $2(n-1)!X_n = (-1)^{n-1} S_{n-1}$.

Integrating the relation

$$(\sin t)^x = \sum_{n=0}^{\infty} \ln^n \sin t \frac{x^n}{n!}$$

produces

$$\frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} S_n = \int_0^{\pi/2} \sin^x t \, dt = \frac{\sqrt{\pi}\,\Gamma(x/2+1/2)}{2\Gamma(x/2+1)}.$$

Differentiate logaritmically and use

$$\psi\left(\tfrac{x}{2} + \tfrac{1}{2}\right) - \psi\left(\tfrac{x}{2} + 1\right) = -2\ln 2 + 2 \sum_{n=1}^{\infty} (-1)^{n+1} (1-2^{-n}) \zeta(n+1) x^n,$$

yields, after matching coefficients, the recurrence

$$S_n = -(\ln 2)\, S_{n-1} - \sum_{r=1}^{n-1} (-1)^r (1-2^{-r}) \frac{(n-1)!\zeta(r+1)}{(n-1-r)!} S_{n-1-r}.$$

The initial value $S_0 = \pi$ now gives all the values S_n. In particular, $S_1 = -\pi \ln 2$ produces the stated value of S_2.