PROOF OF FORMULA 4.225.2

$$\int_0^{\pi/4} \ln(\cos x + \sin x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln(\cos x + \sin x)\, dx = -\frac{\pi}{8}\ln 2 + \frac{G}{2}$$

The change of variables $t = \frac{\pi}{2} - x$ applied to the first integral gives

$$\int_0^{\pi/4} \ln(\cos x + \sin x)\, dx = \int_{\pi/4}^{\pi/2} \ln(\cos x + \sin x)\, dx =$$

and this proves the first identity.

To obtain the second one, observe that

$$\begin{aligned}
\int_0^{\pi/2} \ln(\cos x + \sin x)\, dx &= \int_0^{\pi/2} \ln\left[\sqrt{2}\cos(x - \pi/4)\right]\, dx \\
&= \frac{\pi}{4}\ln 2 + \int_{-\pi/4}^{\pi/4} \ln\cos t\, dt \\
&= \frac{\pi}{4}\ln 2 + 2\int_0^{\pi/4} \ln\cos t\, dt
\end{aligned}$$

and the result follows from entry 4.224.5:

$$\int_0^{\pi/4} \ln\cos t\, dt = -\frac{\pi}{4}\ln 2 + \frac{G}{2}.$$