PROOF OF FORMULA 4.231.2

$$\int_0^1 \frac{\ln x\, dx}{1-x} = -\frac{\pi^2}{6}$$

Let $t = -\ln x$ to obtain

$$\int_0^1 \frac{\ln x\, dx}{1-x} = -\int_0^\infty \frac{t\, dt}{e^t - 1}.$$

The integral representation 9.513.1 of the Riemann zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\, dt}{e^t - 1}$$

now gives

$$\int_0^1 \frac{\ln x\, dx}{1-x} = -\zeta(2) = -\frac{\pi^2}{6}.$$