

PROOF OF FORMULA 4.232.3

$$\int_0^\infty \frac{\ln x \, dx}{(x+a)(x-1)} = \frac{\pi^2 + \ln^2 a}{2(a+1)}$$

Expand the integrand in partial fractions to obtain

$$\int_0^\infty \frac{\ln x \, dx}{(x+a)(x-1)} = \frac{1}{1+a} \left(\lim_{N \rightarrow \infty} I_2(N) - I_3(N) \right)$$

where

$$I_2(N) = \int_0^N \frac{\ln x \, dx}{x-1} \text{ and } I_3(N) = \int_0^N \frac{\ln x \, dx}{x+a}.$$

Split the integral I_2 into two parts: one from 0 to 1 and the second one from 1 to N . Let $t = 1/x$ in the second integral to produce

$$I_2(N) = - \int_0^1 \frac{\ln x \, dx}{1-x} - \int_{1/N}^1 \frac{\ln t}{t} dt - \int_{1/N}^1 \frac{\ln t}{1-t} dt.$$

A similar argument gives

$$I_3(N) = \ln a \ln(1 + N/a) + \int_0^{N/a} \frac{\ln t \, dt}{1+t}.$$

Thus, to evaluate the original integral and after computing the second integral in I_2 , it is required to find the limit as $N \rightarrow \infty$ of the expression

$$- \int_0^1 \frac{\ln x \, dx}{1-x} + \frac{1}{2} \ln^2 N - \int_{1/N}^1 \frac{\ln t \, dt}{1-t} - \ln a \ln(1 + N/a) - \int_0^{N/a} \frac{\ln t \, dt}{1+t}.$$

Separate the last integral on $0 \leq t \leq 1$ and $1 \leq t \leq N$. Then

$$\int_0^\infty \frac{\ln x \, dx}{(x+a)(x-1)} = \frac{1}{1+a} \lim_{N \rightarrow \infty} \left(\frac{5\pi^2}{12} + \frac{\ln^2 N}{2} - \ln a \ln(N+a) + \ln^2 a - \int_1^{N/a} \frac{\ln t \, dt}{1+t} \right)$$

Integrate by parts in the last integral to obtain

$$\int_1^{N/a} \frac{\ln t \, dt}{1+t} = \frac{N}{2(a+N)} \ln^2(a/N) - \frac{\pi^2}{12} + o(1).$$

The result now follows from

$$\lim_{N \rightarrow \infty} \ln(N+a) - \frac{N}{a+N} \ln N = 0.$$